FREE MATH TUTOR

THE QUADRATIC FORMULA & EQUATION : STEP BY STEP

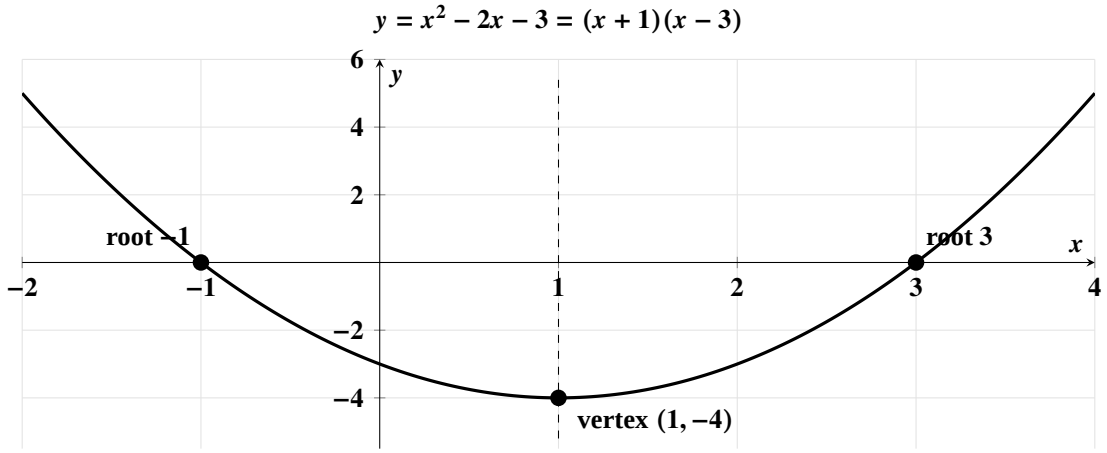
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^2 + bx + c = 0, \quad a \neq 0.$

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

An illustrated textbook from the first definition to the cubic formula

Definitions · Derivations · Worked examples · Practice and solutions



THE CENTRAL IDEA

An equation asks which inputs make a statement true. A formula gives a systematic way to find them. A graph makes the real solutions visible.

With Cardano's cubic formula and an article on higher-degree equations

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Classroom and independent-study edition

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How to use this textbook

Read each definition, cover the worked solution, and attempt the example. Then compare your steps, not just the final number. Use the practice chapter before opening the answer key. Exact answers come first; decimal approximations are labeled with ≈.

Route	Suggested sequence
Core algebra	Chapters 1–4: language, derivation, solving, graphs, and applications.
Extension	Chapter 5: the cubic formula, complex cube roots, and three real roots.
Beyond quadratics	Chapters 6–7: higher-degree equations and numerical root finding.
Independent review	Chapter 8, followed by the worked answer key and formula reference.

Conventions and prerequisites

The main lessons assume real coefficients unless a section says otherwise. Solutions may be sought in $\mathbb{R}$ (real numbers) or $\mathbb{C}$ (complex numbers). A complex number can also be real; "nonreal" means its imaginary part is not zero. You should be able to combine like terms, multiply signed numbers, distribute a factor, and work with fractions.

A *root of an equation* is a solution; a *square root* or *cube root* is a number obtained by reversing a power. The same word serves two related roles. Figures are mathematical illustrations, not photographs. Geometric sketches are schematic unless axes or dimensions give a scale.

Source guide. Bracketed source labels refer to clickable references at the end. Explanations, numerical examples, and illustrations in this edition were created for this textbook; source notes identify references for the standard mathematical results.

CHAPTER 1

The language of quadratic equations

Learning goals

Distinguish an expression, equation, and function; recognize degree; identify signed coefficients; and explain what a solution means.

1.1 What makes an equation quadratic?

A polynomial in one variable is a finite sum of terms made from constants multiplied by nonnegative integer powers of that variable. Its degree is the largest exponent with a nonzero coefficient after like terms have been combined. A quadratic polynomial has degree two.

Standard quadratic form

$$ax^2 + bx + c = 0, \qquad a \neq 0.$$

Here x is the unknown, a is the quadratic coefficient, b is the linear coefficient, and c is the constant term. The restriction $a \neq 0$ is part of the definition, not an optional detail.

Term	Definition and example
Expression	A mathematical combination without an equality claim, such as $2x^2 - 5x - 3$.
Equation	A statement that two expressions are equal, such as $2x^2 - 5x - 3 = 0$.
Function	A rule assigning each allowed input one output: $f(x) = 2x^2 - 5x - 3$.
Variable	A symbol whose value may vary or may be unknown. Here it is x.
Coefficient	The numerical factor multiplying a power. In $-5x$, the coefficient is -5, not 5.
Constant	A term independent of the variable. Here $c = -3$.
Term	A part separated by addition or subtraction; the sign belongs to it.
Root / solution	An input that makes the equation true. A zero of f is an input for which $f(x) = 0$.
Degree	The highest surviving exponent; the degree of $2x^2 - 5x - 3$ is 2.

One polynomial, three viewpoints

Take $f(x) = x^2 - 5x + 6$. As an expression, it can be evaluated or rearranged. As an equation, $x^2 - 5x + 6 = 0$ asks for special inputs. As a graph, $y = f(x)$ shows all real input–output pairs. The real roots are the x-coordinates where that graph meets the horizontal axis.[S1, S2]

$$f(2) = 4 - 10 + 6 = 0, \qquad f(3) = 9 - 15 + 6 = 0.$$

Thus 2 and 3 are roots. The points $(2, 0)$ and $(3, 0)$ are intercepts. A root is a number; an intercept is a point.

1.2 Read coefficients only after standardizing

Worked example 1: terms on both sides

Solve later, but first identify the coefficients in $3x(x - 2) = 5$.

Step 1. **Expand the product: $3x^2 - 6x = 5$.**

Step 2. Subtract 5 from both sides: $3x^2 - 6x - 5 = 0$.

Step 3. Match powers: $a = 3$, $b = -6$, and $c = -5$.

The right side must be zero before the formula is used in its standard form. Reading $c = 5$ from the original right side would put the wrong sign into the formula.

Worked example 2: the degree can change

Consider $x^2 + 4x = x^2 + 9$. Subtract x^2 from each side:

$$4x = 9 \quad \Longrightarrow \quad x = \frac{9}{4}.$$

This is a linear equation after simplification, not a quadratic. Its displayed appearance before cancellation is not its final degree.

Standard form	a	b	c
$x^2 - 7x + 10 = 0$	1	-7	10
$-2x^2 + 3x + 2 = 0$	-2	3	2
$5x^2 - 45 = 0$	5	0	-45
$x^2 + 4x = 0$	1	4	0
$\frac{1}{2}x^2 - \frac{3}{4}x - \frac{1}{2} = 0$	$\frac{1}{2}$	$-\frac{3}{4}$	$-\frac{1}{2}$

Missing terms still have coefficients

If the x term is absent, then $b = 0$. If the constant term is absent, then $c = 0$. Write those zeros before substituting; do not shift the remaining numbers into the wrong slots.

1.3 Equivalent equations and safe operations

Adding the same expression to both sides preserves equality. Multiplying or dividing both sides by a *nonzero constant* preserves the solution set. Multiplying by an expression that can be zero requires more care; dividing by it can discard solutions.

Do not divide away a root

From $x^2 + 4x = 0$, dividing by x would give $x + 4 = 0$, but it would silently exclude $x = 0$. Factor instead:

$$x(x + 4) = 0 \quad \Longrightarrow \quad x = 0 \text{ or } x = -4.$$

The zero-product property says that a product of real or complex numbers is zero exactly when at least one factor is zero.

Quick check

Is $4x^2 = 12$ quadratic? What are its coefficients in standard form?

Check your reasoning: Yes: $4x^2 - 12 = 0$, so $a = 4$, $b = 0$, and $c = -12$. Dividing by 4 gives the equivalent equation $x^2 - 3 = 0$.

CHAPTER 2

The quadratic formula: meaning and proof

Learning goals

Read every part of the formula, interpret the discriminant, derive the formula by completing a square, and choose a suitable solution method.

2.1 The entire formula

Quadratic formula

For $ax^2 + bx + c = 0$ with $a \neq 0$,

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.}$$

The symbol ± directs us to calculate two branches:

$$x_+ = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad x_- = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The whole numerator is divided by $2a$. The entire quantity $b^2 - 4ac$ is under the square-root sign.[S1]

Part	What to do
$-b$	Take the opposite of the signed coefficient b. If $b = -7$, then $-b = 7$.
b^2	Square the whole signed coefficient: $(-7)^2 = 49$.
$4ac$	Multiply 4, a, and c, keeping all signs.
$\sqrt{b^2 - 4ac}$	Evaluate the radicand, then take a square root in the chosen number system.
±	Use the plus branch and the minus branch separately. They coincide for a repeated root.
$2a$	Multiply 2 by a; divide every term in the numerator by this same denominator.

2.2 The discriminant predicts the roots

The *discriminant* of a quadratic is $\Delta = b^2 - 4ac$. For real coefficients, its sign classifies the solutions.[S1]

Condition	Roots in $\mathbb{C}$	Real graph
$\Delta > 0$	Two distinct real roots	Two horizontal-axis intersections
$\Delta = 0$	One real root of multiplicity 2	One point of contact with the axis
$\Delta < 0$	Two distinct nonreal conjugate roots	No horizontal-axis intersections

For *integer* coefficients, a positive perfect-square discriminant gives rational roots. A positive nonsquare discriminant gives irrational roots. These rationality tests require the stated coefficient assumption; the sign test applies to all real coefficients.

A root count is not always a distinct-root count

A quadratic has two complex roots *counting multiplicity*. If it is $(x-2)^2=0$, the root 2 occurs twice, but there is only one distinct solution. Multiplicity records how many times a factor appears.[S4]

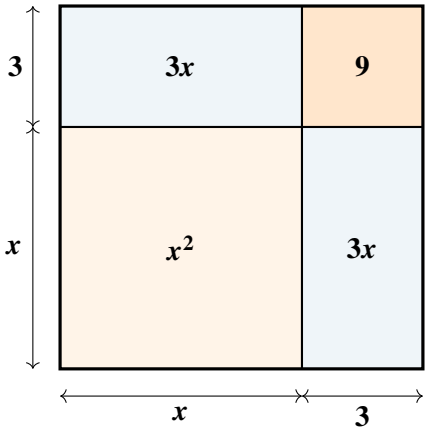
2.3 Why completing a square works

The identity

$$(x+h)^2 = x^2 + 2hx + h^2$$

shows that the coefficient of x is twice the number added inside the parentheses. To turn x^2+6x into a square, add $(6/2)^2=9$:

$$x^2 + 6x + 9 = (x+3)^2.$$



The two strips have total area $6x$.

The missing corner has area $3\times 3=9$.

The completed square has side $x+3$.

Area model shown for $x>0$. The algebraic identity remains true for every real or complex x.

Figure 2.1: Completing the square: two rectangular strips and one corner turn x^2+6x into $(x+3)^2$. The sketch is schematic.

Worked example 3: solve by completing the square

Step 1. Start with $x^2+6x+5=0$, so $x^2+6x=-5$.

Step 2. Add 9 to *both* sides: $x^2+6x+9=4$.

Step 3. Write the square: $(x+3)^2=4$.

Step 4. Use both square roots: $x+3=\pm 2$.

Step 5. Subtract 3: $x=-3\pm 2$, giving $x=-1$ or $x=-5$.

Quick check

What number completes the square in $x^2-10x+\square$?

Check your reasoning: $(-10/2)^2=25$, because $(x-5)^2=x^2-10x+25$.

2.4 Derivation that keeps every sign under control

Here is a complete derivation of the quadratic formula. It avoids taking a square root of a^2 separately, so there is no hidden assumption that a is positive.

Step 1. Start in standard form, with $a\neq 0$:

$$ax^2+bx+c=0.$$

Step 2. Multiply the entire equation by the nonzero number $4a$:

$$4a^2x^2 + 4abx + 4ac = 0.$$

Step 3. Move the constant term:

$$4a^2x^2 + 4abx = -4ac.$$

Step 4. Add b^2 to both sides:

$$4a^2x^2 + 4abx + b^2 = b^2 - 4ac.$$

Step 5. Recognize a perfect square on the left:

$$(2ax + b)^2 = b^2 - 4ac.$$

Step 6. Take both square roots, using complex numbers when necessary:

$$2ax + b = \pm\sqrt{b^2 - 4ac}.$$

Step 7. Subtract b, then divide the entire expression by $2a$:

$$2ax = -b \pm \sqrt{b^2 - 4ac}.$$

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

Each earlier operation was reversible because $a \neq 0$. This is why the formula finds all the solutions and does not invent additional ones. The derivation is an algebraic proof, not merely a pattern noticed in examples.

The principal square root and the two solutions

For real $r > 0$, the notation $\sqrt{r}$ means the nonnegative square root, not "plus or minus." Thus $\sqrt{9} = 3$, whereas solving $u^2 = 9$ gives $u = \pm 3$. The $\pm$ belongs to the solution step. When $r = 0$, those two branches coincide.

2.5 Choose the simplest valid method

Method	Best first choice when	Example
Factorization	Factors are easy to recognize	$x^2 - 5x + 6 = (x - 2)(x - 3)$
Square-root method	A square is isolated	$(x - 1)^2 = 7$
Completing the square	You want vertex form or a derivation	$x^2 + 4x + 1 = (x + 2)^2 - 3$
Quadratic formula	Any genuine quadratic, especially awkward coefficients	$3x^2 - 6x - 5 = 0$
Graph or numerical check	You want to visualize or approximate real roots	Compare intercepts with exact answers

Seven-step routine: Standardize; identify coefficients; compute the discriminant; substitute; simplify; calculate both branches; verify.

CHAPTER 3

Solving quadratics, step by step

Learning goals

Solve quadratics with integer, fractional, irrational, repeated, and nonreal roots. Handle missing terms, negative leading coefficients, and fractions without losing signs.

3.1 Two integer roots

Worked example 4: $x^2 - 5x + 6 = 0$

Step 1. Identify $a = 1$, $b = -5$, and $c = 6$.

Step 2. Calculate the discriminant: $\Delta = (-5)^2 - 4(1)(6) = 25 - 24 = 1$.

Step 3. Substitute into the full formula:

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(6)}}{2(1)} = \frac{5 \pm 1}{2}.$$

Step 4. Separate the branches:

$$x_+ = \frac{6}{2} = 3, \qquad x_- = \frac{4}{2} = 2.$$

Step 5. Check both roots: $9 - 15 + 6 = 0$ and $4 - 10 + 6 = 0$.

$$\boxed{x = 2 \text{ or } x = 3}$$

Another method. The numbers -2 and -3 multiply to 6 and add to -5. Therefore $(x - 2)(x - 3) = 0$, giving the same roots.

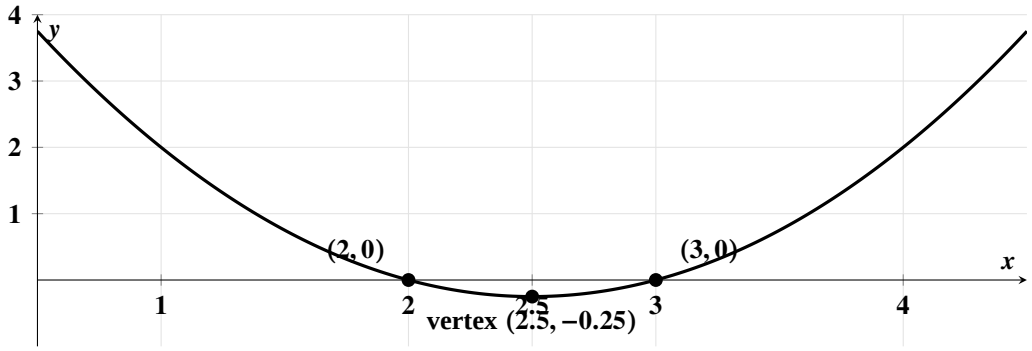


Figure 3.1: The two solutions are the real inputs at the two axis crossings. The vertex lies halfway between them horizontally.

3.2 A fractional root and a nonunit coefficient

Worked example 5: $2x^2 - 5x - 3 = 0$

Step 1. Record $a = 2$, $b = -5$, and $c = -3$.

Step 2. Evaluate $\Delta = 25 - 4(2)(-3) = 25 + 24 = 49$.

Step 3. Use $2a = 4$, not 2:

$$x = \frac{5 \pm \sqrt{49}}{4} = \frac{5 \pm 7}{4}.$$

Step 4. Split the result: $x_+ = 12/4 = 3$ and $x_- = -2/4 = -1/2$.

Step 5. Check the fractional root exactly:

$$2\left(-\frac{1}{2}\right)^2 - 5\left(-\frac{1}{2}\right) - 3 = \frac{1}{2} + \frac{5}{2} - 3 = 0.$$

$$\boxed{x = 3 \text{ or } x = -\frac{1}{2}}$$

The factorization $(2x+1)(x-3) = 0$ confirms both roots. Fractions are exact answers, not unfinished calculations.

3.3 Two irrational roots

Worked example 6: $x^2 + 2x - 1 = 0$

Step 1. Here $a = 1$, $b = 2$, and $c = -1$.

Step 2. Compute $\Delta = 2^2 - 4(1)(-1) = 8$.

Step 3. Apply the formula: $x = (-2 \pm \sqrt{8})/2$.

Step 4. Extract the perfect-square factor: $\sqrt{8} = \sqrt{4 \cdot 2} = 2\sqrt{2}$.

Step 5. Divide the *whole* numerator by 2:

$$x = \frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}.$$

$$\boxed{x = -1 + \sqrt{2} \text{ or } x = -1 - \sqrt{2}}$$

The approximate roots are 0.414214 and −2.414214. The radical answers are exact; the displayed decimals are rounded. Exact check. Write $s = \pm\sqrt{2}$, so $s^2 = 2$. Then

$$(-1 + s)^2 + 2(-1 + s) - 1 = s^2 - 2 = 0.$$

Radical simplification rules

For nonnegative real u, v, $\sqrt{uv} = \sqrt{u}\sqrt{v}$. But square roots do not distribute across addition: $\sqrt{9+16} = 5$, while $\sqrt{9} + \sqrt{16} = 7$. Also $\sqrt{x^2} = |x|$ for real x, not automatically x.

3.4 One repeated real root

Worked example 7: $9x^2 + 12x + 4 = 0$

Step 1. Set $a = 9$, $b = 12$, and $c = 4$.

Step 2. Calculate $\Delta = 144 - 4(9)(4) = 0$.

Step 3. Substitute: $x = (-12 \pm 0)/18 = -2/3$.

Step 4. Recognize the repeated factor: $(3x + 2)^2 = 0$.

$$\boxed{x = -\frac{2}{3} \quad \text{with multiplicity 2}}$$

Both formula branches produce the same number. This is one distinct solution, not two different ones. Substitution gives $4 - 8 + 4 = 0$.

3.5 Two nonreal complex roots

A complex number has the form $u + v\mathrm{i}$, where $u, v \in \mathbb{R}$ and $\mathrm{i}^2 = -1$. The *real part* is u and the *imaginary part* is v. Its conjugate is $u - v\mathrm{i}$.[S3]

Worked example 8: $2x^2 - 4x + 7 = 0$

Step 1. Record $a = 2$, $b = -4$, and $c = 7$.

Step 2. Compute $\Delta = 16 - 56 = -40$. There are no real solutions.

Step 3. Work in $\mathbb{C}$: $\sqrt{-40} = \mathrm{i}\sqrt{40} = 2\mathrm{i}\sqrt{10}$.

Step 4. Substitute and simplify:

$$x = \frac{4 \pm 2\mathrm{i}\sqrt{10}}{4} = 1 \pm \frac{\sqrt{10}}{2}\mathrm{i}.$$

$$\boxed{x = 1 + \frac{\sqrt{10}}{2}\mathrm{i} \quad \text{or} \quad x = 1 - \frac{\sqrt{10}}{2}\mathrm{i}}$$

Step 5. Check both at once. Put $v = \pm\mathrm{i}\sqrt{10}/2$, so $v^2 = -5/2$. Then

$$2(1 + v)^2 - 4(1 + v) + 7 = 2v^2 + 5 = 0.$$

Interpretation. The real graph has no horizontal-axis crossing. That does not mean the equation has no solutions in $\mathbb{C}$. The quadratic formula continues to work when the number system is enlarged.

3.6 An equation that must be rearranged first

Worked example 9: $3x(x - 2) = 5$

Step 1. Expand and move everything left: $3x^2 - 6x - 5 = 0$.

Step 2. Identify $a = 3$, $b = -6$, and $c = -5$.

Step 3. Find $\Delta = 36 - 4(3)(-5) = 96$.

Step 4. Use $\sqrt{96} = 4\sqrt{6}$:

$$x = \frac{6 \pm 4\sqrt{6}}{6} = 1 \pm \frac{2\sqrt{6}}{3}.$$

Step 5. Verify in the original equation. Let $v = \pm 2\sqrt{6}/3$, so $v^2 = 8/3$:

$$3(1 + v)((1 + v) - 2) = 3(1 + v)(-1 + v) = 3(v^2 - 1) = 5.$$

$$\boxed{x = 1 \pm \frac{2\sqrt{6}}{3}}$$

The checks use the original statement, not only the rearranged polynomial. This habit is especially useful in problems involving fractions or later restrictions.

3.7 Fractional coefficients

Worked example 10: $\frac{1}{2}x^2 - \frac{3}{4}x - \frac{1}{2} = 0$

Step 1. Multiply every term by the common denominator 4:

$$2x^2 - 3x - 2 = 0.$$

Step 2. Use the simplified coefficients $a = 2$, $b = -3$, $c = -2$.

Step 3. Find $\Delta = 9 - 4(2)(-2) = 25$.

Step 4. Calculate $x = (3 \pm 5)/4$, giving 2 and $-1/2$.

Step 5. Check in the fraction equation: for $x = 2$, $2 - 3/2 - 1/2 = 0$; for $x = -1/2$, $1/8 + 3/8 - 1/2 = 0$.

$$\boxed{x = 2 \text{ or } x = -\frac{1}{2}}$$

Clearing constant denominators is optional, but often reduces arithmetic. Because $4 \neq 0$, this multiplication preserves every solution.

3.8 Negative leading coefficient and missing terms

Worked example 11: $-2x^2 + 3x + 2 = 0$

Step 1. Use $a = -2$, $b = 3$, and $c = 2$ exactly as written.

Step 2. Compute $\Delta = 9 - 4(-2)(2) = 25$.

Step 3. Remember that $2a = -4$:

$$x = \frac{-3 \pm 5}{-4}.$$

Step 4. Evaluate separately: $(-3 + 5)/(-4) = -1/2$ and $(-3 - 5)/(-4) = 2$.

$$\boxed{x = -\frac{1}{2} \text{ or } x = 2}$$

Alternatively, multiply the original equation by -1 to get $2x^2 - 3x - 2 = 0$. The roots do not change. The labels "plus branch" and "minus branch" do not necessarily mean larger root and smaller root.

Worked example 12: a missing linear term

For $5x^2 - 45 = 0$, the coefficients are $a = 5$, $b = 0$, $c = -45$.

$$\Delta = 0 - 4(5)(-45) = 900, \qquad x = \frac{0 \pm 30}{10} = \pm 3.$$

The faster route is $x^2 = 9$, then $x = \pm 3$. Both methods give exactly the same answer.

Worked example 13: a missing constant term

For $x^2 + 4x = 0$, $a = 1$, $b = 4$, and $c = 0$:

$$x = \frac{-4 \pm \sqrt{16}}{2} = \frac{-4 \pm 4}{2}.$$

Therefore $x = 0$ or $x = -4$. Factoring gives $x(x + 4) = 0$ and is shorter.

Boundary case: what happens when $a = 0$?

The equation becomes $bx + c = 0$. If $b \neq 0$, then $x = -c/b$. If $b = 0$ and $c \neq 0$, there is no solution. If $a = b = c = 0$, every number in the chosen domain is a solution. Never substitute $a = 0$ into the quadratic formula.

3.9 Troubleshooting: six common errors

Error	Correction
Reading a negative coefficient as positive	Write $b = -5$ and then $-b = -(-5) = 5$.
Squaring without parentheses	$(-5)^2 = 25$, but $-5^2 = -25$ under the usual order of operations.
Dividing only the radical by $2a$	Write one fraction bar under the entire numerator.
Dropping one $\pm$ branch	Calculate the plus case and minus case on separate lines.
Calling $\Delta < 0$ "no solutions"	Say "no real solutions"; give the complex roots when working in $\mathbb{C}$.
Rounding before finishing	Simplify exactly first. Round only the final numerical report.

Quick check

A student finds $x = (-6 \pm \sqrt{12})/6$ and writes $x = -1 \pm \sqrt{3}$. Is this correct?

Check your reasoning: No. Since $\sqrt{12} = 2\sqrt{3}$, the answer is $x = -1 \pm \sqrt{3}/3$. The factor 2 in the radical numerator does not cancel the entire denominator 6.

Comparison table: recognize the result before you calculate

Equation	Δ	Roots and classification
$x^2 - 5x + 6 = 0$	1	$2, 3$; two integer roots
$2x^2 - 5x - 3 = 0$	49	$3, -1/2$; two rational roots
$x^2 + 2x - 1 = 0$	8	$-1 \pm \sqrt{2}$; two irrational roots
$9x^2 + 12x + 4 = 0$	0	$-2/3$; repeated real root
$2x^2 - 4x + 7 = 0$	-40	$1 \pm i\sqrt{10}/2$; nonreal pair
$5x^2 - 45 = 0$	900	± 3; missing linear term
$x^2 + 4x = 0$	16	$0, -4$; missing constant term

A discriminant tells you the type of roots before you finish the arithmetic. A final root list should agree with that prediction. This check catches many sign errors without repeating the entire calculation.

CHAPTER 4

Graphs, geometry, and applications

Learning goals

Connect roots to intercepts and the vertex; use the discriminant visually; interpret complex roots correctly; and build quadratic models with meaningful domains and units.

4.1 Three useful forms of one quadratic

For $a \neq 0$, three descriptions emphasize different features:

Name	Form	What it reveals
Standard	$f(x) = ax^2 + bx + c$	Coefficients; vertical intercept $(0, c)$
Vertex	$f(x) = a(x - h)^2 + k$	Vertex (h, k); axis $x = h$
Factored	$f(x) = a(x - r_1)(x - r_2)$	Roots r_1, r_2, possibly complex

Completing the square gives

$$h = -\frac{b}{2a}, \qquad k = c - \frac{b^2}{4a} = -\frac{\Delta}{4a}.$$

For real coefficients, the graph is a parabola. It opens upward when $a > 0$ and downward when $a < 0$. Its vertical axis of symmetry is $x = h$.[S2]

Worked example 14: read the graph from algebra

For $f(x) = x^2 - 2x - 3$, complete the square:

$$f(x) = (x - 1)^2 - 4.$$

Thus the vertex is $(1, -4)$ and the symmetry line is $x = 1$. The factored form is $(x + 1)(x - 3)$, so the real roots are -1 and 3. The vertical intercept is $(0, -3)$.

The formula as a center plus two offsets

$$x = -\frac{b}{2a} \pm \frac{\sqrt{\Delta}}{2a}.$$

The center is the vertex's horizontal coordinate. When $\Delta > 0$, the two real roots lie on opposite sides of that center, each at horizontal distance $\sqrt{\Delta}/(2|a|)$. This distance uses $|a|$, because a distance cannot be negative.

4.2 A visual discriminant comparison

Compare three parabolas with the same shape and the same symmetry line:

$$f(x) = (x - 1)^2 - 4, \quad g(x) = (x - 1)^2, \quad h(x) = (x - 1)^2 + 2.$$

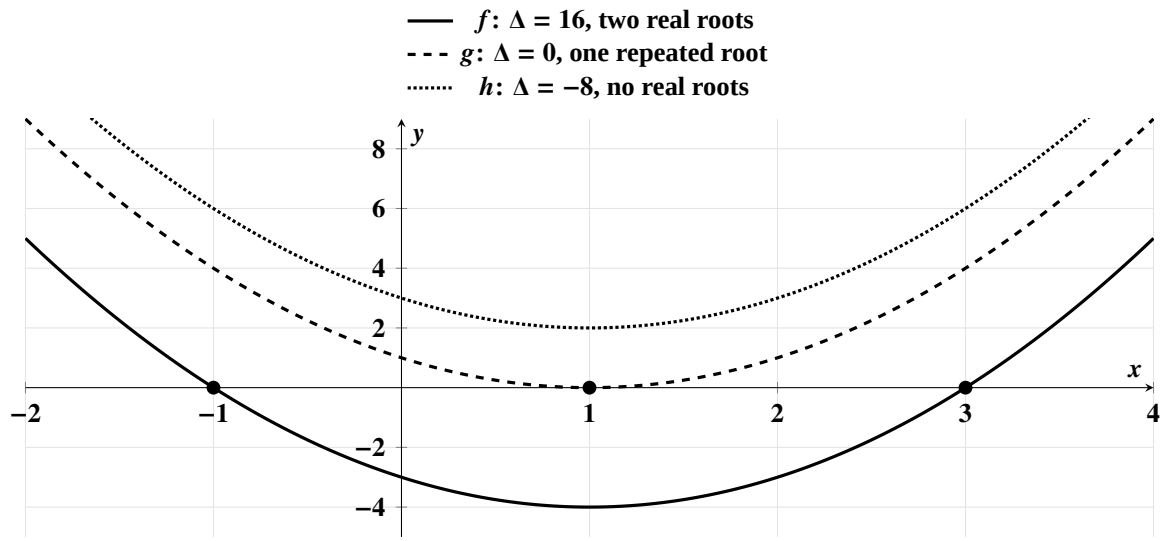


Figure 4.1: Vertical translation changes the discriminant and the real intersections without changing the parabola's opening or symmetry line. Line styles remain distinguishable in grayscale.

For the third curve, $x^2 - 2x + 3 = 0$ has roots $1 \pm i\sqrt{2}$. A real Cartesian graph of $y = f(x)$ does not display nonreal inputs as horizontal-axis crossings.

4.3 Where complex roots can be pictured

In the complex plane, $u + vi$ is represented by the point (u, v). This is a different picture from the graph of the function. It plots *solutions as numbers*, rather than plotting real inputs against outputs.[S3]

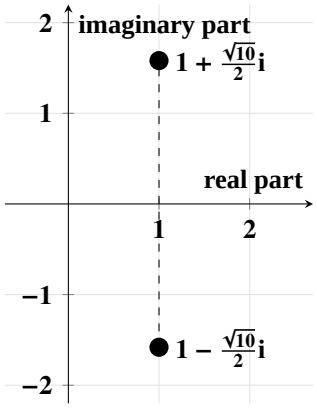


Figure 4.2: The roots of $2x^2 - 4x + 7 = 0$ form a conjugate pair, reflected across the real axis in the complex plane.

A useful check: Viète's relations

Expanding $a(x - r_1)(x - r_2)$ and comparing coefficients gives

$$\boxed{r_1 + r_2 = -\frac{b}{a}, \qquad r_1 r_2 = \frac{c}{a}.}$$

For the complex roots above, the sum is 2 and the product is $1 + 10/4 = 7/2$. These agree with $-b/a = 2$ and $c/a = 7/2$. A matching sum and product give a strong algebraic check.

4.4 Application: dimensions of a rectangle

Worked example 15: area 40 m^2, length 3 m more than width

Let w be the numerical width in meters; the length is $w + 3$ meters.

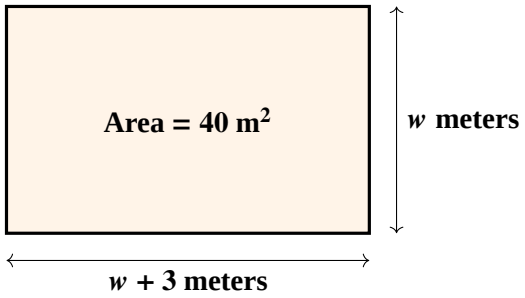


Figure 4.3: A geometric condition becomes an equation: width multiplied by length equals area.

Step 1. Use area = width × length: $w(w+3) = 40$.

Step 2. Write $w^2 + 3w - 40 = 0$, so $a = 1$, $b = 3$, $c = -40$.

Step 3. Calculate $\Delta = 9 + 160 = 169$.

Step 4. Apply the formula: $w = (-3 \pm 13)/2$, so $w = 5$ or $w = -8$.

Step 5. Impose the geometric domain $w > 0$. Only $w = 5$ is a physical width.

$$\boxed{\text{width} = 5 \text{ m}, \qquad \text{length} = 8 \text{ m}}$$

The rejected value $w = -8$ is an algebraic root, but it does not satisfy the measurement condition. A model can have algebraic solutions that do not describe the situation.

4.5 Application: a height–time model

Worked example 16: when does the object reach the ground?

For this exercise, use the idealized model

$$H(t) = -5t^2 + 20t + 5,$$

where t is time in seconds after release and H is height in meters. This is a specified classroom model with constant downward acceleration 10 m/s^2 and no air resistance; it is not a claim that this acceleration is exact everywhere.

Step 1. At ground level $H = 0$: $-5t^2 + 20t + 5 = 0$.

Step 2. Divide by -5: $t^2 - 4t - 1 = 0$.

Step 3. Compute $\Delta = 16 + 4 = 20$.

Step 4. Solve: $t = (4 \pm \sqrt{20})/2 = 2 \pm \sqrt{5}$.

Step 5. Keep the value in the time domain $t \geq 0$:

$$\boxed{t = 2 + \sqrt{5} \approx 4.236 \text{ s}}$$

The negative root $2 - \sqrt{5}$ lies before the model's release time. It is not the future landing time.

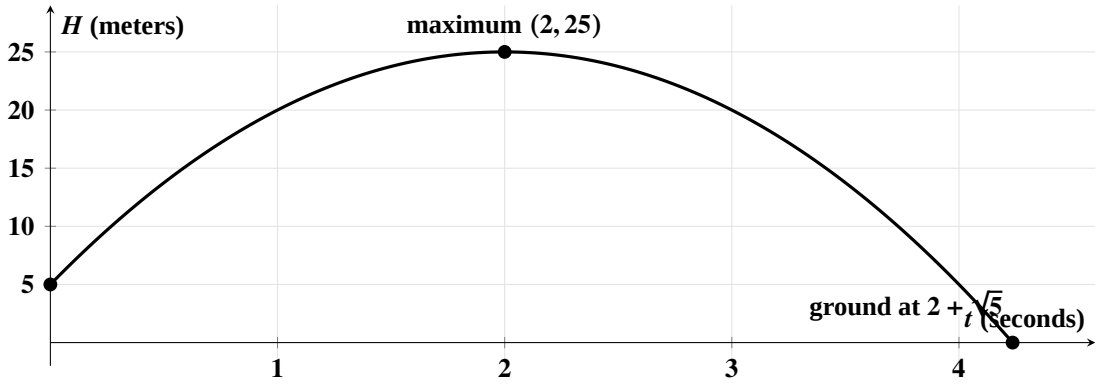


Figure 4.4: The physically used portion of the model begins at release and ends at its positive ground-level root.

Another question from the same model. Complete the square:

$$H(t) = -5(t-2)^2 + 25.$$

The model's maximum height is 25 meters at $t = 2$ seconds. Finding a root answers "when is height zero?" Finding the vertex answers "when is height greatest?"

4.6 Application: intersections of a line and a parabola

Worked example 17: $y = x^2$ and $y = 2x + 3$

At an intersection, the two expressions for y must be equal:

$$x^2 = 2x + 3 \quad \Longrightarrow \quad x^2 - 2x - 3 = 0.$$

Step 1. Use $a = 1$, $b = -2$, $c = -3$, giving $\Delta = 4 + 12 = 16$.

Step 2. Find $x = (2 \pm 4)/2$, so $x = -1$ or $x = 3$.

Step 3. Find each corresponding y: $(-1)^2 = 1$ and $3^2 = 9$.

$$\boxed{(-1, 1) \quad \text{and} \quad (3, 9)}$$

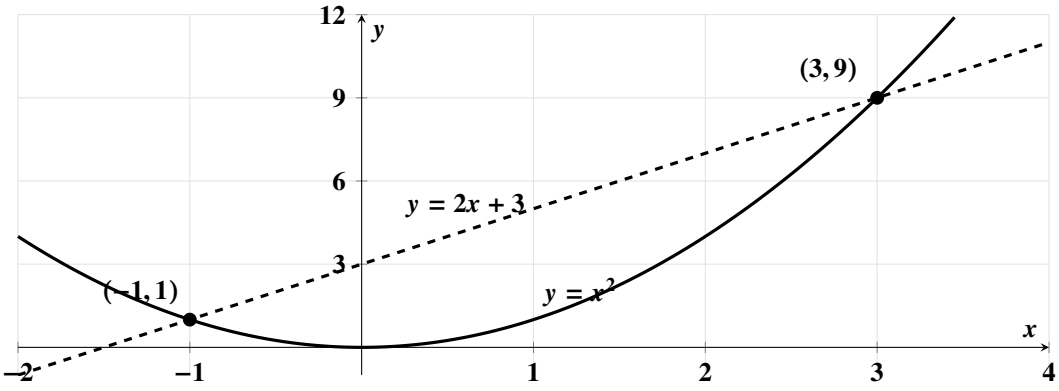


Figure 4.5: Setting two function rules equal turns an intersection problem into a quadratic equation.

Quick check

What does a zero discriminant mean in a line–parabola intersection problem?

Check your reasoning: After subtraction produces a genuine quadratic, $\Delta = 0$ means one repeated real intersection input. The line is tangent to the parabola at that point.

CHAPTER 5

The cubic equation and Cardano's formula

Learning goals

Reduce a cubic to depressed form; derive Cardano's formula using a quadratic; calculate all three roots consistently; and handle repeated and three-real-root cases.

5.1 From degree two to degree three

A cubic equation has the standard form

$$ax^3 + bx^2 + cx + d = 0, \qquad a \neq 0.$$

Here d is the constant term. A cubic has three complex roots counting multiplicity. With real coefficients, at least one root is real. The other two are either real or a nonreal conjugate pair.[S4]

Term	Definition
Monic	A polynomial whose leading coefficient is 1. Divide a cubic by a to make it monic.
Depressed cubic	A cubic with no squared term: $t^3 + pt + q = 0$. "Depressed" describes the removed term.
Cardano's formula	A radical method for cubic roots, built from square roots and cube roots.
Cube root of unity	A number z satisfying $z^3 = 1$. The three such numbers coordinate the three cubic roots.
Branch choice	A choice among the possible complex values of a root operation. Compatible choices are essential.

5.2 Remove the squared term

Divide by a, then substitute

$$x = t - \frac{b}{3a}.$$

After expansion and collection, the t^2 terms cancel. The result is

The depressed-cubic coefficients

$$t^3 + pt + q = 0,$$

$$\boxed{p = \frac{3ac - b^2}{3a^2}, \qquad q = \frac{2b^3 - 9abc + 27a^2d}{27a^3}.}$$

The original variable is recovered by $x = t - b/(3a)$.[S7]

Why this shift? If the monic squared-term coefficient is $B = b/a$, then $(t - B/3)^3$ contributes $-Bt^2$. The term $B(t - B/3)^2$ contributes $+Bt^2$. Their cancellation is the entire purpose of the substitution.

5.3 Deriving the cubic formula from a quadratic

Start with $t^3 + pt + q = 0$ and try $t = u + v$. Expanding gives

$$(u+v)^3 + p(u+v) + q = u^3 + v^3 + (3uv + p)(u+v) + q.$$

Choose u and v so that

$$uv = -\frac{p}{3}, \qquad u^3 + v^3 = -q.$$

The first condition removes the middle term. Define $U = u^3$ and $V = v^3$. Then

$$U + V = -q, \qquad UV = (uv)^3 = -\frac{p^3}{27}.$$

Therefore U and V are the two roots of a *quadratic* equation:

$$z^2 + qz - \frac{p^3}{27} = 0.$$

Apply the quadratic formula:

$$z = \frac{-q \pm \sqrt{q^2 + 4p^3/27}}{2} = -\frac{q}{2} \pm \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}.$$

Define the *Cardano radicand*

$$D_C = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3.$$

Cardano's cubic formula: one root

$$\boxed{x = -\frac{b}{3a} + \sqrt[3]{-\frac{q}{2} + \sqrt{D_C}} + \sqrt[3]{-\frac{q}{2} - \sqrt{D_C}}.}$$

The quantities p, q, and D_C are defined above. The two cube roots must be chosen so that their product is $-p/3$. For real coefficients with $D_C \geq 0$, taking real cube roots gives a real solution. For $D_C < 0$, intermediate complex values require compatible branches.[S7]

What the derivation teaches. A cubic becomes manageable when two auxiliary quantities have a known sum and product. That is exactly the situation a quadratic equation describes. The quadratic formula is therefore part of the machinery behind the cubic formula.

5.4 All three roots, with a safe branch rule

Let

$$\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}, \qquad \omega^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}.$$

These obey $\omega^3 = 1$, $1 + \omega + \omega^2 = 0$, and $\omega\omega^2 = 1$. Choose a nonzero cube root

$$u^3 = -\frac{q}{2} + \sqrt{D_C}, \qquad v = -\frac{p}{3u}.$$

If this cube-root radicand is zero, use the other square-root sign to get a nonzero one. If $p = q = 0$, treat the triple-root case directly instead.

The complete three-root formula

$$\boxed{\begin{aligned} x_0 &= -\frac{b}{3a} + u + v, \\ x_1 &= -\frac{b}{3a} + \omega u + \omega^2 v, \\ x_2 &= -\frac{b}{3a} + \omega^2 u + \omega v. \end{aligned}}$$

Equivalently, $x_k = -b/(3a) + \omega^k u + \omega^{-k} v$ for $k = 0, 1, 2$. Defining $v = -p/(3u)$ enforces the required product and avoids arbitrary, incompatible cube-root selections.

When u and v are real, the last two roots can be written as

$$x_{1,2} = -\frac{b}{3a} - \frac{u+v}{2} \pm \frac{\sqrt{3}}{2}(u-v)\mathrm{i}.$$

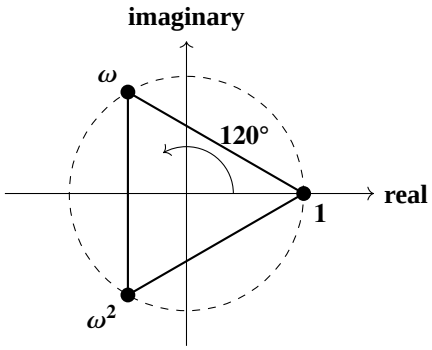


Figure 5.1: The three cube roots of unity are equally spaced on the unit circle. They organize the three compatible combinations in Cardano's formula.

5.5 A coefficient-only version of the cubic formula

Some references package Cardano's formula without writing p and q. For the same cubic $ax^3 + bx^2 + cx + d = 0$, define

$$\Delta_0 = b^2 - 3ac, \qquad \Delta_1 = 2b^3 - 9abc + 27a^2 d.$$

Choose a *nonzero* value

$$C = \sqrt[3]{\frac{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}}.$$

Then all three roots are

Cubic formula directly in the coefficients

$$\boxed{x_k = -\frac{1}{3a}\left(b + \omega^k C + \frac{\Delta_0}{\omega^k C}\right), \quad k = 0, 1, 2.}$$

If the displayed choice gives $C = 0$, reverse the sign before the square root. If both choices give zero, then $\Delta_0 = \Delta_1 = 0$ and the cubic has the triple root $x = -b/(3a)$. This version and the depressed-cubic version are algebraically equivalent.

This is a complete formula only when its definitions and exceptional case are kept with it. Copying the root expression without the meanings of C, Δ_0, Δ_1, and ω leaves an unusable recipe.

5.6 Worked cubic example: one real root and a conjugate pair

Worked example 18: $x^3 - 6x^2 + 18x - 40 = 0$

Step 1. Identify $a = 1$, $b = -6$, $c = 18$, and $d = -40$.

Step 2. Remove the squared term with $x = t + 2$. Calculate

$$p = \frac{3(1)(18) - (-6)^2}{3} = \frac{54 - 36}{3} = 6,$$

$$q = \frac{2(-6)^3 - 9(1)(-6)(18) + 27(1)^2(-40)}{27} = \frac{-432 + 972 - 1080}{27} = -20.$$

The transformed equation is $t^3 + 6t - 20 = 0$.

Step 3. Calculate the Cardano radicand:

$$D_C = (-20/2)^2 + (6/3)^3 = 100 + 8 = 108.$$

Step 4. Take the real cube roots:

$$u = \sqrt[3]{10 + 6\sqrt{3}} = 1 + \sqrt{3}, \qquad v = \sqrt[3]{10 - 6\sqrt{3}} = 1 - \sqrt{3}.$$

These equalities can be checked by cubing. Note that $v < 0$ and $uv = 1 - 3 = -2 = -p/3$.

Step 5. Find the real depressed root $t = u + v = 2$, then undo the shift: $x = t + 2 = 4$.

Step 6. Use the other two branches. Since $u + v = 2$ and $u - v = 2\sqrt{3}$,

$$t_{1,2} = -1 \pm 3\mathrm{i}, \qquad x_{1,2} = 1 \pm 3\mathrm{i}.$$

$$\boxed{x = 4, \quad x = 1 + 3\mathrm{i}, \quad x = 1 - 3\mathrm{i}}$$

Step 7. Check by factoring the original polynomial:

$$(x - 4)(x^2 - 2x + 10) = x^3 - 6x^2 + 18x - 40.$$

The quadratic factor gives $x = (2 \pm \sqrt{-36})/2 = 1 \pm 3\mathrm{i}$. This independently confirms the two nonreal roots.

Three-root consistency check

For a cubic, $r_1 + r_2 + r_3 = -b/a$, $r_1r_2 + r_1r_3 + r_2r_3 = c/a$, and $r_1r_2r_3 = -d/a$. Here these are 6, 18, and 40: the conjugate pair has product 10, while $4((1 + 3\mathrm{i}) + (1 - 3\mathrm{i})) = 8$.

5.7 The cubic discriminant is not the Cardano radicand

For real coefficients, the classification is:

D_C	Roots of the cubic
$D_C > 0$	One real root and two nonreal conjugate roots
$D_C = 0$	Repeated roots; all roots are real
$D_C < 0$	Three distinct real roots; Cardano's radical calculation passes through complex numbers

The usual discriminant of the *original* cubic is

$$\Delta_{\text{cubic}} = b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd = -108a^4D_C.$$

Thus its sign is opposite to the sign of D_C. Authors use different symbols and conventions, so check what a printed "discriminant" means before using a sign table.

Worked example 19: repeated roots

Solve $t^3 - 3t - 2 = 0$. Here $p = -3$, $q = -2$, and

$$D_C = (-1)^2 + (-1)^3 = 0.$$

Take $u = v = 1$. The three-root formula gives

$$t_0 = 2, \qquad t_1 = \omega + \omega^2 = -1, \qquad t_2 = -1.$$

Indeed,

$$t^3 - 3t - 2 = (t - 2)(t + 1)^2.$$

The root 2 is simple, while -1 has multiplicity 2.

The triple-root case

If $p = q = 0$, then $t^3 = 0$, so $t = 0$ has multiplicity 3. For example, $(x + 2)^3 = 0$ has only the distinct root $x = -2$, counted three times. Do not use a branch formula that divides by $u = 0$.

5.8 Three real roots through trigonometry

When $D_C < 0$, one can keep the calculation real by using a trigonometric substitution. This is especially useful for a cubic with rational coefficients that is irreducible over $\mathbb{Q}$ and has three real roots, traditionally associated with the *casus irreducibilis*.[S8] Since $D_C < 0$ implies $p < 0$, put

$$t = 2\sqrt{-\frac{p}{3}}\cos\theta.$$

The identity $4\cos^3\theta - 3\cos\theta = \cos(3\theta)$ turns the cubic into

$$\cos(3\theta) = \frac{-q}{2(-p/3)^{3/2}}.$$

Choose

$$\theta = \frac{1}{3}\arccos\!\left(\frac{-q}{2(-p/3)^{3/2}}\right).$$

Then the three roots of the original cubic are

Real trigonometric form for $D_C < 0$

$$\boxed{x_k = -\frac{b}{3a} + 2\sqrt{-\frac{p}{3}}\cos\!\left(\theta + \frac{2\pi k}{3}\right), \quad k = 0, 1, 2.}$$

Use radians with this displayed formula. Equivalently, add $120^\circ k$ when all the angle calculations are in degrees.

Worked example 20: $t^3 - 3t + 1 = 0$

Step 1. Here $p = -3$, $q = 1$, and $D_C = 1/4 - 1 = -3/4 < 0$.

Step 2. The amplitude is $2\sqrt{-p/3} = 2$.

Step 3. Solve $\cos(3\theta) = -1/2$. Choose $3\theta = 120^\circ$, so $\theta = 40^\circ$.

Step 4. Use the three angles 40°, 160°, and 280°:

$$\boxed{t = 2\cos 40^\circ, \quad 2\cos 160^\circ, \quad 2\cos 280^\circ.}$$

Numerically, the roots are approximately 1.532089, −1.879385, and 0.347296. Their sum is zero, as required by the missing squared term.

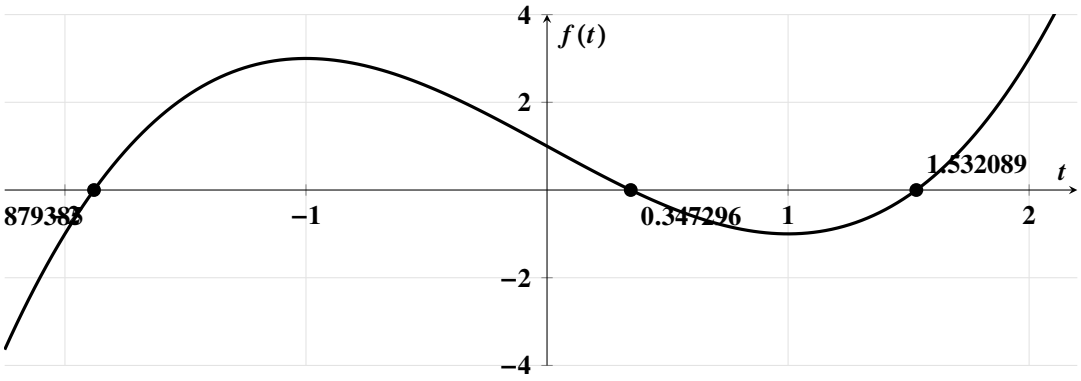


Figure 5.2: All three roots are real even though the square root in Cardano's radical form is nonreal. Intermediate complex values do not force the final roots to be nonreal.

Exact verification. For any of the selected angles α, 3α has cosine $-1/2$. Therefore $(2\cos\alpha)^3 - 3(2\cos\alpha) + 1 = 2\cos(3\alpha) + 1 = 0$.

Cubic method-selection guide

Feature after shifting	Efficient next step
$p = q = 0$	Solve $t^3 = 0$ directly; the root is triple.
$q = 0$	Factor $t(t^2 + p) = 0$; then solve the quadratic factor.
An obvious rational root	Confirm it, divide out its linear factor, and solve the remaining quadratic.
$D_C > 0$	Real cube roots in Cardano's formula produce the real root; obtain the conjugate pair next.
$D_C = 0$	Expect repeated roots; report their multiplicities.
$D_C < 0$	Use the cosine method or carefully coordinated complex cube-root branches.

Two fast cubic checks

For $t^3 - 4t = 0$, factoring gives $t(t^2 - 4) = 0$, so the roots are $0, 2, -2$. No cube-root arithmetic is necessary. For $t^3 + 4t = 0$, the same method gives $0, 2i, -2i$. Both cubics have three complex roots counting multiplicity, but different numbers of real roots.

CHAPTER 6

Article: the world beyond quadratic equations

Learning goals

Understand what higher degree changes, what it does not change, why a general radical formula stops at degree four, and how special structure still makes many higher-degree equations solvable.

6.1 Higher-degree polynomials: more roots, richer structure

A polynomial equation of degree n has the form

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0, \qquad a_n \neq 0.$$

"Higher-order polynomial" is a common phrase, but *higher-degree polynomial* is more precise. In this article, it means degree three or above. A cubic has degree three; a quartic, four; a quintic, five; and a sextic, six.

Increasing the degree enlarges the range of possible behavior, but it does not guarantee that every example becomes harder. The equation $x^{10} - 1 = 0$ has a highly organized set of roots, while $x^3 - 3x + 1 = 0$ already requires more care than simple rational factorization. Degree sets the overall framework; coefficient patterns determine how much of that framework a particular equation uses.

A crucial distinction is between *existence*, *expression*, and *computation*. Existence asks whether roots are there. Expression asks which symbolic language can describe them. Computation asks how to obtain useful numerical values. These are different tasks. A theorem about one should not be mistaken for a theorem about all three.

The fundamental theorem of algebra

Every nonconstant polynomial with complex coefficients has exactly as many complex roots as its degree, *counting multiplicity*. Real numbers are included among complex numbers. A degree-n polynomial has at most n distinct real roots.[S4]

For example,

$$P(x) = (x-2)^3(x+1)^2$$

has degree five, but only two distinct roots: 2 and −1. Their multiplicities, 3 and 2, add to five. A root count that forgets multiplicity would misread both the algebra and the graph.

6.2 What graphs can reveal, and what they can hide

A real polynomial graph is continuous and smooth. At a real root of odd multiplicity, its sign changes and the graph crosses the axis. At a real root of even multiplicity, its sign does not change and the graph touches the axis without crossing. Higher multiplicities can make the contact flatter.[S5]

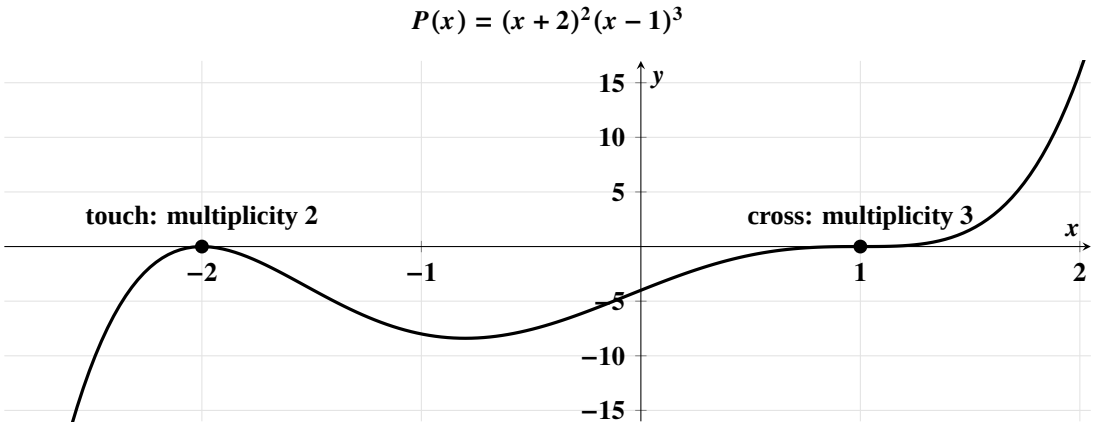


Figure 6.1: A fifth-degree polynomial with only two distinct real roots. The root multiplicities explain the different behavior at each intercept.

A degree-n polynomial has at most $n - 1$ turning points, not necessarily exactly $n - 1$. An odd-degree real polynomial has opposite end behavior and therefore at least one real root. An even-degree real polynomial can have no real roots at all; $x^4 + 1 = 0$ is a simple example.[S5]

A graphing window, however, is not a proof that all roots have been found. Two crossings can be very close together; a root can lie outside the chosen window; a repeated root may not produce a sign change; and nonreal roots do not appear as real-axis intersections. A trustworthy solution combines graphical intuition with algebraic structure or certified numerical reasoning.

6.3 Why the quadratic formula does not simply grow forever

The quadratic formula uses a finite combination of arithmetic operations and square roots. Cardano's formula adds cube roots. General quartic equations can also be solved using radicals, often by introducing an auxiliary cubic. That progression naturally suggests looking for a similar universal formula for every higher degree.

The Abel–Ruffini theorem gives a precise limit: there is no general formula in finitely many arithmetic operations and radicals that solves every polynomial equation of degree five or higher. The words *general*, *finite*, and *radicals* are essential. The theorem does not say the roots fail to exist, that numerical approximation is impossible, or that every individual quintic resists radicals.[S9]

For example, $x^5 - 32 = 0$ has the real solution $x = 2$, and all its complex roots can be written using roots of unity. A polynomial already given as a product of linear and quadratic factors is also solvable, regardless of its total degree. Special coefficient relationships can make a high-degree equation far easier than an unstructured lower-degree one.

Galois theory explains the radical-solvability question through the symmetries of a polynomial's roots. Very roughly, it asks whether those symmetries can be dismantled by the kinds of extensions produced by taking radicals. This is a structural criterion, not a claim that humans merely have not searched hard enough for a long formula.[S9]

The practical lesson is not to abandon higher-degree equations. It is to choose an appropriate language: a factorization, an exact root defined by a polynomial and an isolating interval, a trigonometric expression, or a numerical approximation with an error statement. An exact number need not fit into a short expression containing square-root symbols.

Degree	Name	General radical situation
1	Linear	Solved by arithmetic, when the leading coefficient is nonzero
2	Quadratic	Quadratic formula
3	Cubic	Cardano's formula, with compatible root choices
4	Quartic	Radical methods exist; an auxiliary cubic is often used
5	Quintic	No universal radical formula; special cases can be solvable
$n \geq 5$	Higher degree	No universal radical formula for arbitrary coefficients; structural and numerical methods remain useful

6.4 Structure first: factors, substitutions, and symmetry

Before applying a large formula, inspect the equation. Is every term divisible by x? Are only even powers present? Do exponents appear in a regular pattern? Is there an obvious root such as 1 or -1? A successful substitution can lower the degree of the *equation being solved* even though the original polynomial still has its original degree.

For a polynomial with integer coefficients and nonzero constant term, the rational-root theorem provides a finite candidate list: a rational root r/s in lowest terms must have r dividing the constant coefficient and s dividing the leading coefficient. The list contains candidates, not guaranteed roots. Testing and division must still follow.[S4]

Once a root r is confirmed, the factor theorem guarantees a factor $x - r$. Polynomial division removes that factor and lowers the remaining degree by one. The division algorithm writes a polynomial as divisor times quotient plus remainder; substituting $x = r$ shows why division by $x - r$ leaves remainder $P(r)$.[S6]

Symmetry also matters. A polynomial containing only even powers satisfies $P(-x) = P(x)$, so its roots occur in opposite pairs. A real-coefficient polynomial has nonreal roots in conjugate pairs. These observations provide both shortcuts and error checks: a claimed answer list that violates the required pairing is incomplete or incorrect.

6.5 Worked higher-degree example: a quartic in disguise

Worked example 21: $x^4 - 5x^2 + 4 = 0$

Step 1. Only x^4, x^2, and a constant occur. Set $y = x^2$.

Step 2. Solve the quadratic $y^2 - 5y + 4 = 0$:

$$y = \frac{5 \pm \sqrt{25 - 16}}{2} = \frac{5 \pm 3}{2} = 4 \text{ or } 1.$$

Step 3. Undo the substitution for *each* value: $x^2 = 4$ gives $x = \pm 2$; $x^2 = 1$ gives $x = \pm 1$.

$$\boxed{x = -2, -1, 1, 2}$$

Step 4. Verify the complete factorization:

$$x^4 - 5x^2 + 4 = (x^2 - 4)(x^2 - 1) = (x - 2)(x + 2)(x - 1)(x + 1).$$

The quadratic substitution gave two y values, but the original quartic has four roots. Stopping at $y = 4, 1$ would answer the wrong variable's equation.

Worked example 22: a quartic with nonreal roots

For $x^4 + x^2 - 6 = 0$, set $y = x^2$ to get

$$y^2 + y - 6 = (y + 3)(y - 2) = 0.$$

Thus $y = 2$ or $y = -3$. Returning to x gives

$$\boxed{x = \pm\sqrt{2}, \qquad x = \pm i\sqrt{3}}$$

If the problem asks only for real roots, report $\pm\sqrt{2}$ and explain why the other branch is excluded from $\mathbb{R}$. If it asks for all complex roots, keep all four.

6.6 How a general quartic reduces to quadratics

This optional extension explains the mechanism rather than requiring memorization of one enormous formula. Divide a general quartic by its leading coefficient and write

$$x^4 + Ax^3 + Bx^2 + Cx + D = 0.$$

The shift $x = y - A/4$ removes the cubic term, leaving

$$y^4 + Py^2 + Qy + R = 0,$$

where

$$P = B - \frac{3A^2}{8}, \qquad Q = C - \frac{AB}{2} + \frac{A^3}{8},$$

$$R = D - \frac{AC}{4} + \frac{A^2B}{16} - \frac{3A^4}{256}.$$

For a parameter m, rearrange the depressed quartic as

$$(y^2 + m)^2 = (2m - P)y^2 - Qy + (m^2 - R).$$

We want the right side to be a square $(\alpha y + \beta)^2$. Match coefficients:

$$\alpha^2 = 2m - P, \qquad 2\alpha\beta = -Q, \qquad \beta^2 = m^2 - R.$$

Choose m from the auxiliary cubic

$$\boxed{(2m - P)(m^2 - R) = \frac{Q^2}{4}.}$$

When $Q \neq 0$, such a choice gives $\alpha \neq 0$, so take $\alpha = \sqrt{2m - P}$ and $\beta = -Q/(2\alpha)$. Then solve the two quadratics

$$y^2 - \alpha y + (m - \beta) = 0, \qquad y^2 + \alpha y + (m + \beta) = 0.$$

Finally return to $x = y - A/4$. When $Q = 0$, solve directly as a biquadratic in y^2. This completing-square strategy is a version of Ferrari's method.[S7]

Worked example 23: see the factorization appear

For $y^4 - 5y^2 - 2y + 3 = 0$, use $P = -5$, $Q = -2$, $R = 3$. The value $m = -2$ satisfies the auxiliary cubic because $(1)(1) = (-2)^2/4$. Therefore $\alpha = 1$ and $\beta = 1$, giving

$$(y^2 - 2)^2 = (y + 1)^2.$$

The two quadratics are $y^2 - y - 3 = 0$ and $y^2 + y - 1 = 0$. Hence

$$\boxed{y = \frac{1 \pm \sqrt{13}}{2}, \qquad y = \frac{-1 \pm \sqrt{5}}{2}}$$

This example shows why a quartic solution can contain a cubic step and then two quadratic steps.

6.7 A sextic whose six roots can all be organized

Worked example 24: $x^6 - 7x^3 - 8 = 0$

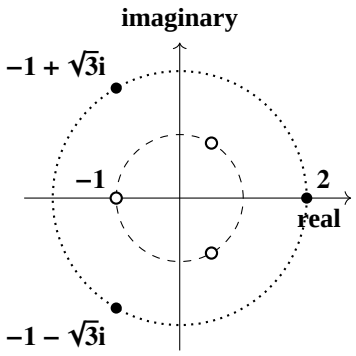
Set $y = x^3$. Then $y^2 - 7y - 8 = (y - 8)(y + 1) = 0$, so $x^3 = 8$ or $x^3 = -1$. The two *real* roots are 2 and -1. Using the cube roots of unity gives all six complex roots:

$$\boxed{2,\ 2\omega,\ 2\omega^2,\ -1,\ -\omega,\ -\omega^2.}$$

In rectangular form, these are

$$2,\ -1 + \sqrt{3}\mathrm{i},\ -1 - \sqrt{3}\mathrm{i},\ -1,\ \frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i},\ \frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}.$$

All six are distinct. The high degree did not prevent an exact solution; the repeated exponent pattern made the original equation quadratic in x^3.



Filled points: $x^3 = 8$; each has modulus 2.

Open points: $x^3 = -1$; each has modulus 1.

Within each group, successive roots are separated by 120°.

Figure 6.2: The six sextic roots split into two triangular groups in the complex plane. Modulus is the distance of a complex number from the origin.

6.8 Why higher-degree equations remain useful

Polynomial models let a solver express more than a constant rate or a single bend. A simple dimension problem can produce a cubic: a box with edge lengths x, $x + 1$, and $x + 2$ and volume 24 gives

$$x(x + 1)(x + 2) = 24 \quad \Longrightarrow \quad x^3 + 3x^2 + 2x - 24 = 0.$$

Testing $x = 2$ gives the physically meaningful dimensions 2, 3, and 4. A more elaborate model may not have a convenient integer root, but the modeling process is the same: define variables, build the relation, solve, and enforce the physical domain.

The central habit is to separate a model from its mathematical solution. Increasing the degree does not automatically make a model more accurate. Every extra coefficient needs justification, and every computed root needs interpretation. Equations are tools for describing relationships; the useful answer is the root that fits both the mathematics and the situation.

Article takeaway

Higher degree means a larger possible root structure, not automatic impossibility. Look for factors and substitutions first. Know the difference between a radical formula and an exact definition. Use approximation deliberately, and verify that a reported root satisfies both the equation and its domain.

CHAPTER 7

Numerical roots and verification

Learning goals

Approximate a root by bisection and Newton's method, distinguish a small residual from a small root error, and recognize when numerical arithmetic can lose accuracy.

7.1 Exact and approximate are different descriptions

The equation $x^5 - x - 1 = 0$ is a useful setting for numerical work. Define

$$f(x) = x^5 - x - 1.$$

Because $f(1) = -1$ and $f(2) = 29$, continuity guarantees a real root between 1 and 2. On that interval, $f'(x) = 5x^4 - 1 > 0$, so the root there is unique. One exact way to name it is "the unique root of $x^5 - x - 1$ in $(1, 2)$." A decimal representation is a separate approximation to that exact number.

7.2 Bisection: a root stays inside a shrinking bracket

For a continuous real function, opposite signs at the endpoints guarantee at least one root between them. Bisection repeatedly tests the midpoint and retains a sign-changing half. It is reliable when a valid sign-changing bracket is available.

Step 1. **Start with $[L, U] = [1, 2]$.**

Step 2. **Compute the midpoint $m = (L + U)/2$.**

Step 3. **Evaluate $f(m)$. If it is exactly zero, stop. Otherwise keep $[L, m]$ or $[m, U]$, whichever has opposite endpoint signs.**

Step 4. **Repeat until the bracket is narrow enough for the required error.**

Step	Tested midpoint	$f(m)$, approximately	Retained bracket
1	1.500000	5.093750	[1, 1.5]
2	1.250000	0.801758	[1, 1.25]
3	1.125000	−0.322968	[1.125, 1.25]
4	1.187500	0.173892	[1.125, 1.1875]
5	1.156250	−0.089639	[1.15625, 1.1875]
6	1.171875	0.038197	[1.15625, 1.171875]

If the initial width is W, after N bisections the retained bracket has width $W/2^N$. The midpoint of that retained bracket is within $W/2^{N+1}$ of a root contained in it. With $W = 1$, nine bisections give a midpoint error at most $1/1024 < 0.001$.

What a sign test does not guarantee

A repeated even-multiplicity root may have the same sign on both sides, so a sign-changing bracket need not exist around it. For example, $(x - 1)^2$ has a root at 1 but never becomes negative. Failure to find a sign change is not proof that no real root exists.

7.3 Newton's method: follow the tangent

At a trial input x_n, the tangent line to $y = f(x)$ is

$$y = f(x_n) + f'(x_n)(x - x_n).$$

Set the tangent's y value to zero and solve for its horizontal intercept. This gives Newton's iteration:

Newton's root-finding formula

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} \qquad (f'(x_n) \neq 0).$$

Here f' is the derivative, the function giving the tangent slope. Newton's method can converge rapidly near a simple root, but it is not guaranteed to converge from every starting value.[S10]

Worked example 25: approximate the quintic root

For $f(x) = x^5 - x - 1$, use $f'(x) = 5x^4 - 1$ and start with $x_0 = 1.2$:

$$x_{n+1} = x_n - \frac{x_n^5 - x_n - 1}{5x_n^4 - 1}.$$

The first update is

$$x_1 = 1.2 - \frac{0.28832}{9.368} \approx 1.169222886.$$

n	x_n, rounded	$f(x_n)$, approximately
0	1.200000000000	$2.88320000 \times 10^{-1}$
1	1.169222886422	$1.59536991 \times 10^{-2}$
2	1.167311020061	$5.83306540 \times 10^{-5}$
3	1.167303978357	$7.88701659 \times 10^{-10}$
4	1.167303978261	1.44199×10^{-19}

The residuals in the table are evaluated from unrounded iterates; the displayed inputs are rounded. Thus the root is approximately 1.167303978. Additional printed digits are meaningful only when supported by the calculation's accuracy, not merely by a display setting.

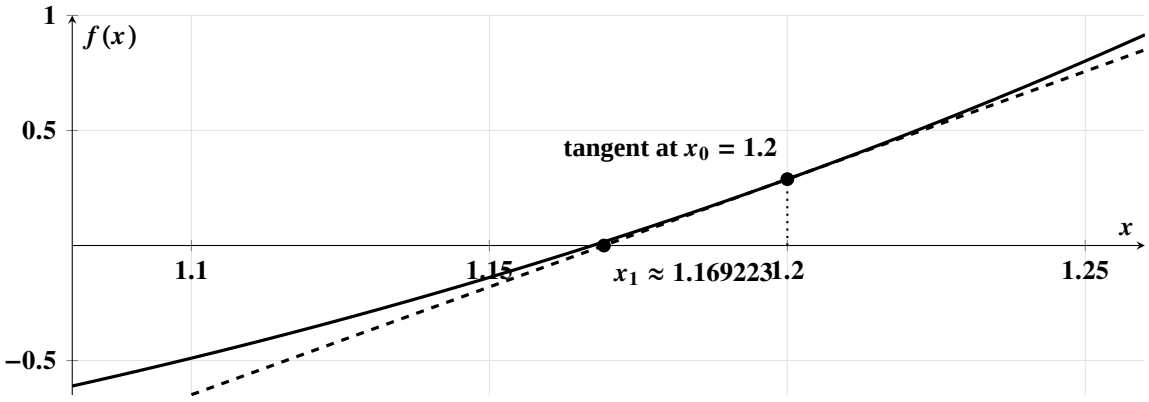


Figure 7.1: The first Newton step replaces the curve locally with its tangent. The tangent's intercept becomes the next trial input.

7.4 Residual, error, and numerical caution

The *residual* of an approximate root $\widehat{x}$ is $f(\widehat{x})$. The *root error* is the difference between $\widehat{x}$ and the true root. They are not the same quantity.

For $f(x) = (x-1)^2$, the approximation $\widehat{x} = 1.01$ has residual 0.0001, but its error is 0.01. A flat graph can make the residual look extremely small while the input is less accurate than expected. A reliable error bound needs more information, such as a bracket or a lower bound on the magnitude of the derivative near a simple root.

Newton's method can fail when the derivative is zero, jump outside a useful interval, or cycle. For instance, applying it to $f(x) = x^3 - 2x + 2$ at $x_0 = 0$ gives $x_1 = 1$, and the next update gives $x_2 = 0$. That two-cycle does not converge to a root. A bracketed method is a useful safeguard.[S10]

A subtle issue even the quadratic formula can face

The formula is exact algebra, but rounded arithmetic may lose significant digits when nearly equal numbers are subtracted. Consider

$$x^2 + 10^8 x + 1 = 0.$$

One root is close to -10^8 and the other is close to -10^{-8}. The expression $(-10^8 + \sqrt{10^{16} - 4})/2$ subtracts two nearly equal large quantities, which can badly damage a floating-point approximation of the small root.

For real coefficients and $\Delta \geq 0$, a more stable evaluation often begins with

$$Q_s = -\frac{1}{2}\left(b + \operatorname{sgn}_+(b)\sqrt{\Delta}\right),$$

where $\operatorname{sgn}_+(b) = 1$ for $b \geq 0$ and -1 for $b < 0$. Then use

$$r_1 = \frac{Q_s}{a}, \qquad r_2 = \frac{c}{Q_s}.$$

The second relation uses the exact root product $r_1 r_2 = c/a$. If $Q_s = 0$, do not divide by it; in this setting $b = c = 0$ and the root is 0 twice. This rearrangement reduces cancellation but does not remove every possible floating-point issue, such as overflow from extreme coefficients.

Turning a residual into a genuine error bound

For the quintic $f(x) = x^5 - x - 1$ on $[1, 2]$, the derivative satisfies $f'(x) = 5x^4 - 1 \geq 4$. Let r be the root in that interval and let $\widehat{x}$ also lie there. By the mean value theorem, for some ξ between r and $\widehat{x}$,

$$f(\widehat{x}) - f(r) = f'(\xi)(\widehat{x} - r).$$

Since $f(r) = 0$ and $|f'(\xi)| \geq 4$, it follows that

$$\boxed{|\widehat{x} - r| \leq \frac{|f(\widehat{x})|}{4}.}$$

Here a small residual *does* give an error bound, because a known slope bound supplies the missing information. The same argument would not work across the flat root of $(x-1)^2$, where the derivative can be zero.

Verification checklist

Check the original equation; count multiplicities; confirm conjugate or opposite-pair structure when applicable; compare sums and products; enforce the physical domain; and distinguish exact equalities from approximate numerical statements.

CHAPTER 8

Practice and assessment

Learning goals

Demonstrate both procedure and reasoning. Give exact answers unless a decimal is requested. Show coefficients, discriminants, and both formula branches for quadratic-formula exercises.

8.1 Foundations: problems 1–6

1. **Put $4x(x+2)=3$ in standard form and identify a, b, c.**
2. **Identify the missing coefficient in $7x^2 - 28 = 0$. Solve it.**
3. **Simplify $x^2 + 5x = x^2 + 10$. Explain why the result is not quadratic.**
4. **Define "root," "degree," "discriminant," and "multiplicity" in your own words.**
5. **Without solving, classify the roots of $3x^2 + 2x + 5 = 0$ using its discriminant.**
6. **Explain the difference between $\sqrt{25}$ and the solutions of $x^2 = 25$.**

8.2 Quadratic equations: problems 7–18

For each equation, solve in $\mathbb{C}$. Identify which roots are real. Verify at least one root by substitution.

7. $x^2 - 7x + 12 = 0$.
8. $3x^2 + 2x - 1 = 0$.
9. $x^2 - 4x - 1 = 0$.
10. $4x^2 - 12x + 9 = 0$.
11. $x^2 + 6x + 13 = 0$.
12. $2x(x-3) = 5$.
13. $-3x^2 + 5x + 2 = 0$.
14. $\frac{1}{2}x^2 + \frac{3}{2}x - 2 = 0$.
15. $5x^2 + 15x = 0$.
16. $2x^2 + 8 = 0$.
17. $x^2 + 2x - 7 = 0$.
18. $6x^2 - x - 2 = 0$.

Self-assessment before checking the answers

Did you move every term to one side? Did you keep the sign of each coefficient? Is every numerator divided by $2a$? Did you distinguish a repeated root from two distinct roots? Is your final answer in the original variable?

8.3 Graphs and models: problems 19–24

19. **For $f(x) = x^2 - 6x + 5$, find the vertex, axis of symmetry, real roots, and vertical intercept. Write both vertex and factored forms.**
20. **For $f(x) = 2x^2 + 4x + 5$, compute the vertex and discriminant. Explain why there is no real-axis crossing. Find the complex roots.**
21. **A rectangle has area 54 m^2, and its length is 3 m greater than its width. Find its dimensions and explain any rejected root.**
22. **In the classroom model $H(t) = -5t^2 + 15t + 20$, height is in meters and time is in seconds after release. Find the future ground time and the maximum height for the modeled flight.**
23. **Find all intersection points of $y = x^2 + 1$ and $y = 3x + 5$.**

24. Construct a quadratic with roots -2 and 5 and leading coefficient 3. Verify the root sum and product from its coefficients.

8.4 Cubic equations: problems 25–30

25. For $x^3 - 3x^2 + 6x - 8 = 0$, calculate the shift, p, q, and D_C. Use Cardano's formula to find a real root, then find all three roots.

26. Solve $t^3 - 6t - 9 = 0$ by Cardano's formula and report all complex roots.

27. Solve $t^3 - 12t + 16 = 0$ and state each root's multiplicity.

28. For $t^3 - 3t + 1 = 0$, derive the three trigonometric roots and approximate each to six decimal places.

29. Explain why taking arbitrary complex cube roots independently can make Cardano's formula fail. State the product condition that must hold.

30. Solve $x^3 - 6x^2 + 12x - 8 = 0$. Explain why a formula containing division by a zero auxiliary quantity should not be used here.

8.5 Higher degree and numerical reasoning: problems 31–36

31. Solve $x^4 - 10x^2 + 9 = 0$ in $\mathbb{C}$.

32. Solve $x^4 + 2x^2 - 8 = 0$ in $\mathbb{C}$, distinguishing real and nonreal roots.

33. Solve $x^6 - 7x^3 - 8 = 0$. Give all six complex roots, not just the real ones.

34. Show that $x = 1$ is a root of $x^3 - 4x^2 + x + 2 = 0$. Divide by $x - 1$ and solve the remaining quadratic.

35. For $f(x) = x^5 - x - 1$, begin with $[1, 2]$. Perform the first three bisections. Then calculate one Newton step from $x_0 = 1.2$.

36. Evaluate the statement: "Every quintic has five distinct real roots, and none can be solved exactly." Correct each false part.

8.6 Discussion and classroom investigation

Choose $a = 1$ and $b = -2$ while allowing c to vary. Study $x^2 - 2x + c = 0$. Explain, without graphing every example separately, when the equation has two distinct real roots, one repeated root, or two nonreal roots. Predict the vertex for every c, and then test your prediction at $c = -3$, $c = 1$, and $c = 3$.

Teacher discussion guide. Here $\Delta = 4 - 4c$ and $f(x) = (x - 1)^2 + c - 1$. The transition occurs at $c = 1$. Students should connect the sign change of the discriminant to the vertex passing through the horizontal axis. The point is to coordinate symbolic, graphical, and verbal explanations, not merely to calculate three answers.

Worked answer key

Foundations: 1–6

1. Expand: $4x^2 + 8x = 3$. Standard form is $4x^2 + 8x - 3 = 0$, so $a = 4$, $b = 8$, and $c = -3$.

2. The missing linear coefficient is $b = 0$. Divide by 7: $x^2 - 4 = 0$. Thus $x = \pm 2$. Each gives $7(4) - 28 = 0$.

3. Subtract x^2 from both sides: $5x = 10$, so $x = 2$. The quadratic terms cancel, leaving a degree-one equation.

4. A root is a value that satisfies the equation. Degree is the highest nonzero-coefficient exponent after simplification. The quadratic discriminant is $b^2 - 4ac$. A root's multiplicity is the number of copies of its linear factor in a factorization.

5. $\Delta = 2^2 - 4(3)(5) = 4 - 60 = -56 < 0$. There are two distinct nonreal conjugate roots and no real roots.

6. $\sqrt{25} = 5$ is the principal square root. The equation $x^2 = 25$ has two solutions, $x = 5$ and $x = -5$.

Quadratic equations: 7–12

7. $a = 1$, $b = -7$, $c = 12$; $\Delta = 49 - 48 = 1$.

$$x = \frac{7 \pm 1}{2} = 4 \text{ or } 3.$$

Both are real. Check: $3^2 - 7(3) + 12 = 9 - 21 + 12 = 0$.

8. $a = 3$, $b = 2$, $c = -1$; $\Delta = 4 + 12 = 16$.

$$x = \frac{-2 \pm 4}{6} = \frac{1}{3} \text{ or } -1.$$

Both are real. Check: $3(1/9) + 2/3 - 1 = 0$.

9. $a = 1$, $b = -4$, $c = -1$; $\Delta = 16 + 4 = 20$.

$$x = \frac{4 \pm 2\sqrt{5}}{2} = 2 \pm \sqrt{5}.$$

Both are real and irrational. With $s = \pm\sqrt{5}$, substitution gives $(2 + s)^2 - 4(2 + s) - 1 = s^2 - 5 = 0$.

10. $a = 4$, $b = -12$, $c = 9$; $\Delta = 144 - 144 = 0$.

$$x = \frac{12 \pm 0}{8} = \frac{3}{2} \quad \text{(multiplicity 2).}$$

The factorization is $(2x - 3)^2$. Check: $4(9/4) - 12(3/2) + 9 = 0$.

11. $a = 1$, $b = 6$, $c = 13$; $\Delta = 36 - 52 = -16$.

$$x = \frac{-6 \pm 4\mathrm{i}}{2} = -3 \pm 2\mathrm{i}.$$

Both are nonreal. Let $s = \pm 2\mathrm{i}$; then $s^2 = -4$, and $(-3 + s)^2 + 6(-3 + s) + 13 = s^2 + 4 = 0$.

12. Expand and standardize: $2x^2 - 6x - 5 = 0$. Thus $\Delta = 36 + 40 = 76 = 4 \cdot 19$.

$$x = \frac{6 \pm 2\sqrt{19}}{4} = \frac{3 \pm \sqrt{19}}{2}.$$

Both are real. With $x = (3 + s)/2$ and $s^2 = 19$, $2x(x - 3) = (s^2 - 9)/2 = 5$, as required.

Quadratic equations: 13–18

13. $a = -3$, $b = 5$, $c = 2$; $\Delta = 25 + 24 = 49$.

$$x = \frac{-5 \pm 7}{-6} = -\frac{1}{3} \text{ or } 2.$$

Both are real. At $x = 2$, the original left side is $-12 + 10 + 2 = 0$.

14. Multiply the entire equation by 2: $x^2 + 3x - 4 = 0$. Then $\Delta = 9 + 16 = 25$.

$$x = \frac{-3 \pm 5}{2} = 1 \text{ or } -4.$$

Both are real. At $x = 1$, the original left side is $1/2 + 3/2 - 2 = 0$.

15. $a = 5$, $b = 15$, $c = 0$; $\Delta = 225$.

$$x = \frac{-15 \pm 15}{10} = 0 \text{ or } -3.$$

Both are real. Factoring gives $5x(x + 3) = 0$; substitution of either value makes a factor zero.

16. $a = 2$, $b = 0$, $c = 8$; $\Delta = -64$.

$$x = \frac{0 \pm 8\mathrm{i}}{4} = \pm 2\mathrm{i}.$$

Both are nonreal. Check: $2(2\mathrm{i})^2 + 8 = 2(-4) + 8 = 0$.

17. $a = 1$, $b = 2$, $c = -7$; $\Delta = 4 + 28 = 32$.

$$x = \frac{-2 \pm 4\sqrt{2}}{2} = -1 \pm 2\sqrt{2}.$$

Both are real. With $s = \pm 2\sqrt{2}$, substitution gives $(-1 + s)^2 + 2(-1 + s) - 7 = s^2 - 8 = 0$.

18. $a = 6$, $b = -1$, $c = -2$; $\Delta = 1 + 48 = 49$.

$$x = \frac{1 \pm 7}{12} = \frac{2}{3} \text{ or } -\frac{1}{2}.$$

Both are real. Check: $6(4/9) - 2/3 - 2 = 8/3 - 2/3 - 2 = 0$.

Graphs and models: 19–24

19. Complete the square and factor:

$$f(x) = (x - 3)^2 - 4 = (x - 1)(x - 5).$$

The vertex is $(3, -4)$; the symmetry line is $x = 3$; the real roots are 1 and 5; and the vertical intercept is $(0, 5)$. The root midpoint $(1 + 5)/2 = 3$ matches the symmetry line.

20. $f(x) = 2(x + 1)^2 + 3$, so the vertex is $(-1, 3)$ and the graph opens upward. Its minimum is above the axis. Also $\Delta = 16 - 40 = -24$. The roots are

$$x = \frac{-4 \pm \mathrm{i}\sqrt{24}}{4} = -1 \pm \frac{\sqrt{6}}{2}\mathrm{i}.$$

This agrees with the absence of real crossings.

21. Let width be w meters. Then $w(w + 3) = 54$, giving $w^2 + 3w - 54 = 0$. The discriminant is $9 + 216 = 225$, so $w = (-3 \pm 15)/2 = 6$ or -9. Reject the negative width. The dimensions are 6 m by 9 m; their product is 54 m^2.

22. Ground level requires $-5t^2 + 15t + 20 = 0$, equivalent to $t^2 - 3t - 4 = 0$. Thus $t = (3 \pm 5)/2 = 4$ or -1. The future ground time is 4 seconds. Completing the square gives

$$H(t) = -5(t - 1.5)^2 + 31.25.$$

The maximum on the modeled flight occurs at $t = 1.5$ s and is 31.25 m.

23. Equate the two expressions: $x^2 + 1 = 3x + 5$, so $x^2 - 3x - 4 = 0$. Then $x = (3 \pm 5)/2 = 4$ or -1. The corresponding heights are 17 and 2. The intersections are $(4, 17)$ and $(-1, 2)$.

24. Build the polynomial from its roots:

$$3(x + 2)(x - 5) = 3(x^2 - 3x - 10) = 3x^2 - 9x - 30.$$

Thus one required equation is $3x^2 - 9x - 30 = 0$. The root sum is $-2 + 5 = 3 = -b/a$, and the product is $-10 = c/a$.

Cubic equations: 25–30

25. $a = 1$, $b = -3$, $c = 6$, $d = -8$, so the shift is $x = t + 1$. Calculate

$$p = (18 - 9)/3 = 3, \qquad q = (-54 + 162 - 216)/27 = -4.$$

Then $D_C = (-2)^2 + 1^3 = 5$. The real depressed root is

$$t = \sqrt[3]{2 + \sqrt{5}} + \sqrt[3]{2 - \sqrt{5}} = 1.$$

To verify the last equality, use

$$u = \frac{1 + \sqrt{5}}{2}, \qquad v = \frac{1 - \sqrt{5}}{2}; \qquad u^3 = 2 + \sqrt{5}, \quad v^3 = 2 - \sqrt{5}.$$

Thus $x = 2$. Divide the original polynomial by $x - 2$:

$$x^3 - 3x^2 + 6x - 8 = (x - 2)(x^2 - x + 4).$$

The full root list is

$$\boxed{x = 2, \qquad x = \frac{1 + \mathrm{i}\sqrt{15}}{2}, \qquad x = \frac{1 - \mathrm{i}\sqrt{15}}{2}.}$$

26. $p = -6$, $q = -9$, so $D_C = 81/4 - 8 = 49/4$. The real cube-root inputs are $9/2 + 7/2 = 8$ and $9/2 - 7/2 = 1$. Take $u = 2$, $v = 1$; then $uv = 2 = -p/3$. Thus $t = 3$ is the real root. The other two are

$$t = -\frac{3}{2} \pm \frac{\sqrt{3}}{2}\mathrm{i}.$$

The factorization $(t - 3)(t^2 + 3t + 3)$ verifies all three.

27. $p = -12$, $q = 16$, so $D_C = 64 - 64 = 0$. Take $u = v = -2$ to get the simple root $t = -4$. The other two values are $-2(\omega + \omega^2) = 2$. Thus

$$t^3 - 12t + 16 = (t + 4)(t - 2)^2,$$

and the roots are -4 with multiplicity 1 and 2 with multiplicity 2.

28. $D_C = -3/4$. Put $t = 2\cos\theta$, obtaining $2\cos(3\theta) + 1 = 0$. Choose $\theta = 40°$, and add $120°$ twice. The roots are

$$2\cos 40° \approx 1.532089, \quad 2\cos 160° \approx -1.879385, \quad 2\cos 280° \approx 0.347296.$$

29. The derivation cancels $(3uv + p)(u + v)$ by requiring $uv = -p/3$. Arbitrarily chosen cube roots may fail that condition even if their cubes are correct. Choose one nonzero u and define $v = -p/(3u)$; use the coordinated powers of ω for the other roots.

30. The equation is $(x - 2)^3 = 0$. The root is $x = 2$ with multiplicity 3. The depressed equation is $t^3 = 0$, with $p = q = 0$. Solve it directly; an auxiliary formula involving division by $u = 0$ is undefined.

Higher degree and numerical reasoning: 31–36

31. Set $y = x^2$. Then $y^2 - 10y + 9 = 0$, so

$$y = \frac{10 \pm \sqrt{100 - 36}}{2} = \frac{10 \pm 8}{2} = 9 \text{ or } 1.$$

Return to x: the four roots are $\boxed{x = \pm 3, \pm 1}$, all real.

32. Set $y = x^2$. Then $y^2 + 2y - 8 = (y + 4)(y - 2) = 0$. Thus $x^2 = 2$ or $x^2 = -4$. The four roots are $\boxed{x = \pm\sqrt{2}, \pm 2\mathrm{i}}$. Only $\pm\sqrt{2}$ are real.

33. Set $y = x^3$. Then $y^2 - 7y - 8 = (y - 8)(y + 1) = 0$. From $x^3 = 8$ get $2, 2\omega, 2\omega^2$; from $x^3 = -1$ get $-1, -\omega, -\omega^2$. All six are roots because cubing any member of each group gives its required value.

34. Evaluate $P(1) = 1 - 4 + 1 + 2 = 0$, confirming the factor $x - 1$. Division gives

$$P(x) = (x - 1)(x^2 - 3x - 2).$$

The quadratic has discriminant $9 + 8 = 17$. Therefore

$$\boxed{x = 1, \qquad x = \frac{3 + \sqrt{17}}{2}, \qquad x = \frac{3 - \sqrt{17}}{2}.}$$

The division can be checked by multiplying the factors back out.

35. The first three midpoint tests are 1.5 (positive), 1.25 (positive), and 1.125 (negative). The retained brackets are $[1, 1.5]$, $[1, 1.25]$, and $[1.125, 1.25]$. One Newton step gives

$$x_1 = 1.2 - \frac{1.2^5 - 1.2 - 1}{5(1.2)^4 - 1} = 1.2 - \frac{0.28832}{9.368} \approx 1.169222886.$$

The two methods produce different kinds of information: a guaranteed bracket versus an improved trial value.

36. A quintic has five *complex* roots *counting multiplicity*, not necessarily five distinct real roots. A real-coefficient quintic has at least one real root. Some quintics have exact radical solutions; for example, $x^5 - 32 = 0$ has real root 2 and four further complex roots. Abel–Ruffini rules out a *general finite radical formula*, not exact definitions or numerical solutions.

Answer-key extension: are any roots missing?

A root list is complete when its linear factors, including their multiplicities, account for the polynomial's entire degree. Counting only the distinct real roots is not sufficient. The practice set contains all of the following patterns:

Problem	Degree	Distinct real roots	Distinct nonreal roots	Total with multiplicity
10	2	1	0	2
11	2	0	2	2
25	3	1	2	3
27	3	2	0	3
31	4	4	0	4
32	4	2	2	4
33	6	2	4	6

Factor check for problem 31. The four real roots yield

$$(x - 3)(x + 3)(x - 1)(x + 1) = (x^2 - 9)(x^2 - 1) = x^4 - 10x^2 + 9.$$

Factor check for problem 32. The nonreal pair is just as necessary as the real pair:

$$(x - \sqrt{2})(x + \sqrt{2})(x - 2\mathrm{i})(x + 2\mathrm{i}) = (x^2 - 2)(x^2 + 4) = x^4 + 2x^2 - 8.$$

Coefficient check for problem 26. Let the nonreal roots be $z = -3/2 + \mathrm{i}\sqrt{3}/2$ and $\bar{z} = -3/2 - \mathrm{i}\sqrt{3}/2$. Then $z + \bar{z} = -3$ and $z\bar{z} = 3$. With the remaining root 3, the root sum is 0, the pairwise-product sum is $3(-3) + 3 = -6$, and the product is 9. These recover every coefficient of $t^3 - 6t - 9$.

What a graph can and cannot certify. The two real crossings in problem 33 locate $x = -1$ and $x = 2$, but do not display the other four roots. A real graph answers the real-intersection question; complex factorization answers the complete-root question. Problem 27 also reminds us that a tangency can represent a repeated root rather than two distinct intersections.

Formula reference and glossary

Quadratic formula reference

Purpose	Formula and condition
Standard form	$ax^2 + bx + c = 0,\ a \neq 0$
Discriminant	$\Delta = b^2 - 4ac$
Both roots	$x = \dfrac{-b \pm \sqrt{\Delta}}{2a}$
Vertex	$h = -\dfrac{b}{2a},\ k = f(h) = -\dfrac{\Delta}{4a}$
Vertex form	$f(x) = a(x - h)^2 + k$
Root sum and product	$r_1 + r_2 = -b/a,\ r_1 r_2 = c/a$
Factored form	$f(x) = a(x - r_1)(x - r_2)$, allowing complex roots
Root distance from center	$\sqrt{\Delta}/(2\lvert a\rvert)$ when $\Delta \geq 0$ and coefficients are real

Cubic formula reference

$$ax^3 + bx^2 + cx + d = 0, \quad a \neq 0, \qquad x = t - \frac{b}{3a}.$$

$$p = \frac{3ac - b^2}{3a^2}, \qquad q = \frac{2b^3 - 9abc + 27a^2d}{27a^3}, \qquad D_C = \frac{q^2}{4} + \frac{p^3}{27}.$$

$$u^3 = -\frac{q}{2} + \sqrt{D_C}, \qquad v = -\frac{p}{3u}, \qquad \omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}.$$

$$\boxed{x_k = -\frac{b}{3a} + \omega^k u + \omega^{-k} v, \quad k = 0, 1, 2.}$$

Use a nonzero u; reverse the square-root sign if needed. If $p = q = 0$, the triple root is $x = -b/(3a)$. For real coefficients, $D_C > 0$ gives one real root, $D_C = 0$ gives repeated real roots, and $D_C < 0$ gives three distinct real roots.

Polynomial and numerical reference

$$P(x) = (x - r)Q(x) + P(r), \qquad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

A degree-n nonconstant polynomial has n complex roots counting multiplicity. After N bisections from width W, the retained bracket has width $W/2^N$, and its midpoint has error at most $W/2^{N+1}$ relative to a contained root.

Glossary

Term	Meaning
Algebraic number	A real or complex number satisfying a nonzero polynomial with integer coefficients.
Axis of symmetry	A line across which a figure reflects onto itself; for a quadratic graph, $x = -b/(2a)$.
Biquadratic	A polynomial or equation involving powers x^4, x^2, and x^0, reducible by $y = x^2$.
Bisection	A root-bracketing method that repeatedly halves a sign-changing interval.
Branch	One selected value among the possible values of a multivalued inverse operation.
Coefficient	A constant multiplier of a variable's power.
Complex conjugates	The pair $u + v\mathrm{i}$ and $u - v\mathrm{i}$ for real u, v.
Complex number	A number $u + v\mathrm{i}$, where u, v are real and $\mathrm{i}^2 = -1$.
Completing the square	Rewriting a quadratic expression as a squared binomial plus a constant.
Constant term	The term containing no variable factor; equivalently the coefficient of x^0.
Cubic	Degree three.
Degree	The largest exponent with a nonzero coefficient after combining like terms.
Depressed cubic	A cubic of the form $t^3 + pt + q$, with its squared term absent.
Derivative	The function describing instantaneous rate of change or tangent slope.
Discriminant	A coefficient expression that detects repeated roots and, in low degrees, helps classify root types.
Equation	A statement asserting equality between two expressions.
Exact value	A value given without rounding, such as $\sqrt{2}$ or an unambiguous algebraic-root definition.
Factor theorem	$P(r) = 0$ exactly when $x - r$ is a factor of $P(x)$.
Imaginary unit	The number i satisfying $\mathrm{i}^2 = -1$.
Irreducible over $\mathbb{Q}$	Not factorizable into lower-degree nonconstant polynomials with rational coefficients.
Irrational real number	A real number not expressible as a ratio of integers.
Leading coefficient	The coefficient of the highest-degree term.
Monic	Having leading coefficient 1.
Multiplicity	The exponent of a root's linear factor in the polynomial's factorization.
Newton's method	Iteratively replacing an input with the horizontal intercept of the tangent at that input.
Parabola	The conic curve that appears as the graph of a real quadratic function.
Polynomial	A finite sum of constant multiples of nonnegative integer powers of a variable.
Quadratic	Degree two.
Quartic / quintic / sextic	Degree four / five / six.
Radical	An expression involving a root operation, such as a square root or cube root.
Radicand	The expression inside a root symbol.
Rational number	A number m/n with integers m, n and $n \neq 0$.
Residual	The value $P(\widehat{x})$ obtained by substituting an approximate root.
Root / zero	An input that makes the polynomial equal zero.
Root of unity	A complex number z satisfying $z^n = 1$ for some positive integer n.
Simple root	A root of multiplicity 1.
Vertex	The turning point of a real quadratic graph.
Viète's relations	Identities connecting sums and products of roots to polynomial coefficients.
Zero-product property	A product is zero exactly when at least one factor is zero, for real or complex numbers.

Sources and further study

These references support the standard formulas and theorems discussed in the text. Click a reference title to open its source. The worked problems, their numerical choices, and the diagrams are original to this edition; source textbook pages and illustrations are not reproduced here. Sources were consulted September 28, 2026.

[S1] **OpenStax. *College Algebra 2e*, section 2.5, "Quadratic Equations." Reference for solution methods and discriminant classification.**

[S2] **OpenStax. *College Algebra 2e*, section 5.1, "Quadratic Functions." Reference for vertex form, symmetry, and graph interpretation.**

[S3] **OpenStax. *College Algebra 2e*, section 2.4, "Complex Numbers." Reference for the imaginary unit, complex arithmetic, and conjugation.**

[S4] **OpenStax. *College Algebra 2e*, section 5.5, "Zeros of Polynomial Functions." Reference for the fundamental theorem of algebra, rational-root testing, and root structure.**

[S5] **OpenStax. *College Algebra 2e*, section 5.3, "Graphs of Polynomial Functions." Reference for end behavior, multiplicity, and turning points.**

[S6] **OpenStax. *College Algebra 2e*, section 5.4, "Dividing Polynomials." Reference for polynomial division and the remainder principle.**

[S7] **National Institute of Standards and Technology. *Digital Library of Mathematical Functions*, section 1.11, "Zeros of Polynomials." In particular, subsection (iii) gives quadratic, cubic, and quartic solution relations. Notation in this textbook has been normalized to $ax^3 + bx^2 + cx + d$.**

[S8] **National Institute of Standards and Technology. *Digital Library of Mathematical Functions*, section 4.43, "Cubic Equations." Reference for trigonometric and hyperbolic approaches to cubic roots. This textbook derives the cosine version directly from the triple-angle identity.**

[S9] **Paul Ramond (2020). *Abel–Ruffini's Theorem: Complex but Not Complicated!* arXiv:2011.05162. A pedagogical exposition of the limitation on general radical formulas and its relationship to root symmetries.**

[S10] **OpenStax. *Calculus Volume 1*, section 4.9, "Newton's Method." Reference for the tangent iteration and possible convergence failures.**

Reading next

After mastering the quadratic and cubic methods, useful next topics are polynomial division, complex numbers in polar form, derivatives, numerical error analysis, and an introduction to groups and Galois theory. These extend the same core question: how can the structure of an equation guide a trustworthy solution?

Topic locator and final review

Topic	Where to study
Coefficients, degree, and standard form	Chapter 1, page 3
Discriminant and formula derivation	Chapter 2, page 5
Step-by-step quadratic examples	Chapter 3, page 8
Graphs, vertex, and applications	Chapter 4, page 13
Cardano and all three cubic roots	Chapter 5, page 17
Higher-degree article and quartics	Chapter 6, page 23
Bisection and Newton's method	Chapter 7, page 28
Practice problems	Chapter 8, page 31
Worked solutions	Page 33
Formula reference and glossary	Page 37
Sources	Page 39

The complete quadratic story in one chain

$$
\begin{aligned}
& ax^2 + bx + c = 0 \quad (a \neq 0) \\
\iff \quad & (2ax + b)^2 = b^2 - 4ac \\
\iff \quad & 2ax + b = \pm\sqrt{b^2 - 4ac} \\
\iff \quad & \boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}.
\end{aligned}
$$

Five questions to ask before declaring a solution complete

1. Have I solved the original equation, not just a transformed variable's equation?
2. Have I included every root required by the number system and counted repeated roots correctly?
3. Do my exact answers, approximate answers, and graph tell a consistent story?
4. Have I enforced every restriction from a denominator, measurement, or time domain?
5. Can I verify the result by substitution, factorization, a root relation, or a numerical bracket?

Final principle

A formula is most powerful when you understand where it comes from, what each symbol means, and how to verify its output. The quadratic formula is not only a shortcut for degree two. It is a bridge to complex numbers, cubic equations, higher-degree structure, and numerical problem solving.