

THE UNIT CIRCLE : BY HAND

From right triangles to exact values, inverse angles, unit vectors, and Euler's identity.

READ THE TRIANGLE. LOCATE THE POINT. RECOVER THE ANGLE.

32 numbered worked examples with 32 matching diagrams.

Exact methods, controlled approximations, exercises, and solutions.

THE CENTRAL RELATIONSHIP

$$x^2 + y^2 = 1$$

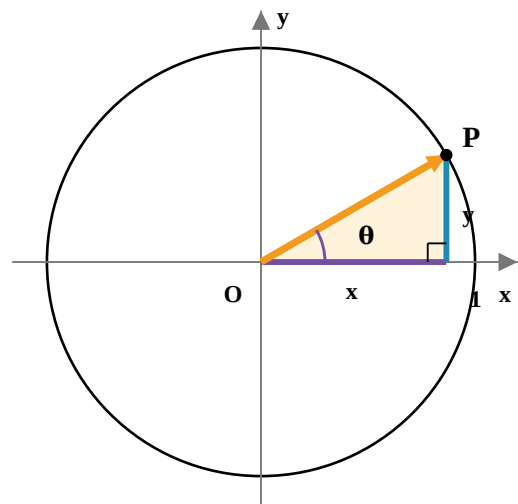
$$\mathbf{u} = (\cos \theta, \sin \theta)$$

$$\sin \theta = y$$

$$\cos \theta = x$$

$$\tan \theta = y/x \quad (x \neq 0)$$

$$P = (\cos \theta, \sin \theta)$$



$$OP = 1$$

$$e^{i\pi} + 1 = 0$$

Angles in degrees, radians, angular minutes, and angular seconds. Every example connects the angle to the arrowhead coordinates and the corresponding unit vector.

Classroom edition • Geometry → Trigonometry → Complex numbers
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A consistent record for every example

Follow the numbered steps, then read Figure NN beside Example NN. Numbering continues through every chapter. The angle record gives the same rotation in four units. The three function values determine $P = (\cos \theta, \sin \theta)$, which is also the component pair of the unit vector from the origin.

Reading key: “=” means exact equality; “ $\approx$ ” marks an approximation. Unless an approximation sign appears, radicals, fractions, and inverse expressions are exact. Angular minutes and seconds are arc units. Work the student exercises on separate paper.

Foundations | words, symbols, and purpose

The central question

Given a right triangle containing an acute angle θ , how do we find $\sin \theta$, $\cos \theta$, $\tan \theta$, and the angle from its ratios? The unit circle organizes the answer and extends it to all directed angles. For an angle outside the acute range, its reference triangle contains the acute reference angle α .

TERM OR SYMBOL	MEANING
θ (theta)	The angle being studied; its unit must be stated.
Hypotenuse	The side opposite the right angle; the longest side.
Opposite / adjacent	Legs named relative to the chosen acute angle.
Unit circle	The circle centered at (0,0), with radius 1.
$P = (x,y)$	The head of the arrow from the origin.
$u = OP$	The vector whose tail is O and head is P. Its length is 1.
$\sin, \cos, \tan$	Ratios or coordinate functions; their outputs are numbers.
$\arcsin, \arccos, \arctan$	Principal inverse functions; their outputs are angles.
π	The ratio of circumference to diameter; one turn is 2π rad.
i and e	$i^2 = -1$; e is the base of the natural exponential.

RIGHT TRIANGLE $\rightarrow$ UNIT CIRCLE

$$\sin \theta = \text{opposite/hypotenuse} = y$$

$$\cos \theta = \text{adjacent/hypotenuse} = x$$

$$\tan \theta = \text{opposite/adjacent} = y/x, \text{ provided } x \neq 0$$

$$\sin^2 \theta + \cos^2 \theta = 1$$

Tools and prerequisites

Use paper, a pencil, fraction arithmetic, the Pythagorean theorem, and square-root simplification. A straightedge and compass help construct exact triangles. Diagrams are guides to relationships; use the labeled mathematics, rather than measuring a printed picture, to justify exact values.

Chapter 1 | Geometry, angle units, and the unit vector

The same direction in three languages

A right triangle gives ratios, a circle gives coordinates, and a vector gives a direction with a length. Put the angle vertex at $O = (0,0)$, align its initial side with the positive x-axis, and let the arrow point toward P. The arrowhead is the point P; the vector OP is the directed displacement from O to P.

Normalize before reading a unit point

For a nonzero vector $v = (a,b)$, first compute $r = \sqrt{a^2+b^2}$, then divide both coordinates by r . The unit vector is $u = (a/r, b/r)$. Its head lies on $x^2+y^2 = 1$. In an acute right triangle, $\cos \theta = \text{adjacent/hypotenuse}$, $\sin \theta = \text{opposite/hypotenuse}$, and $\tan \theta = \text{opposite/adjacent}$.

Angular minutes and seconds

One degree contains 60 angular minutes; one angular minute contains 60 angular seconds. The symbols are $1'$ and $1''$. Use the words arcminute and arcsecond when confusion with time is possible. A full turn contains $21600'$ or $1296000''$.

Degrees-minutes-seconds (DMS)

For a nonnegative angle, $D^\circ M' S'' = (D + M/60 + S/3600)^\circ$. For a negative angle, put the minus sign in front of the whole DMS magnitude. To reverse the conversion, multiply the fractional degree by 60, then multiply the remaining fractional minute by 60.

ROTATION	DEGREES	RADIANS	MINUTES	SECONDS
Full turn	360°	$2\pi \text{ rad}$	$21600'$	$1296000''$
Half-turn	180°	$\pi \text{ rad}$	$10800'$	$648000''$
Quarter-turn	90°	$\pi/2 \text{ rad}$	$5400'$	$324000''$

Conversion rules: radians = degrees $\times \pi/180$; degrees = radians $\times 180/\pi$; arcminutes = degrees $\times 60$; arcseconds = degrees $\times 3600$. Keep π symbolic for exact work.

Before moving on: draw an acute right triangle, mark θ , and name the sides from that vertex. Rotate the drawing on the paper. The opposite, adjacent, and hypotenuse labels remain tied to θ and the right angle, not to which side looks horizontal on the page.

Example 01 | Turn a 3-4-5 triangle into a unit vector

A right triangle has adjacent side 4, opposite side 3, and hypotenuse 5. Find the three ratios and the head of its unit vector.

1. Place θ at the origin. The adjacent side runs along the positive x-axis.
2. Check the right triangle: $4^2 + 3^2 = 16 + 9 = 25 = 5^2$.
3. Divide every length by 5. The hypotenuse becomes 1; the legs become $4/5$ and $3/5$.
4. Read $\cos \theta$ from the horizontal leg and $\sin \theta$ from the vertical leg. Divide y by x for $\tan \theta$.

$$\theta = \alpha = \arctan(3/4)$$

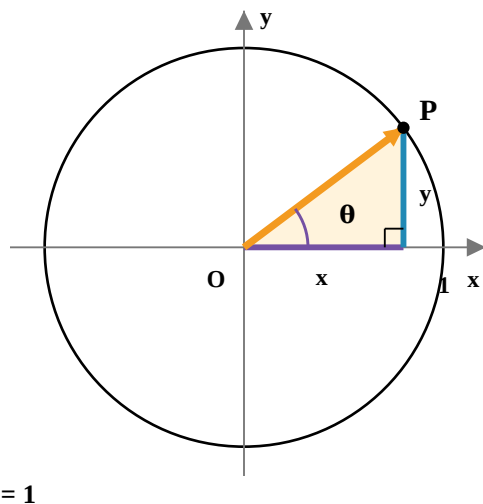


Figure 01. The orange arrow is the unit vector OP ; P is its head. Purple and blue legs show signed x and y projections.

DEGREES	RADIANS	TOTAL MINUTES	TOTAL SECONDS
$180\alpha/\pi$ degrees	α radians	$10800\alpha/\pi$ arcmin	$648000\alpha/\pi$ arcsec
$\sin \theta = y$	$\cos \theta = x$	$\tan \theta = y/x$	
$3/5$	$4/5$	$3/4$	

HEAD COORDINATES: $P = (4/5, 3/5)$

UNIT VECTOR: $u = OP = (4/5, 3/5)$; $\|u\| = 1$.

INVERSE CHECK. $\alpha = \arcsin(3/5) = \arccos(4/5) = \arctan(3/4)$, with $0 < \alpha < \pi/2$.

KEY IDEA. A named inverse value is exact even when it is not a familiar multiple of π . Do not invent a special-angle value.

VERIFY BY HAND. $(4/5)^2 + (3/5)^2 = 25/25 = 1$. Similar triangles have the same ratios.

Example 02 | One angle in four units

Express 30° in radians, total angular minutes, and total angular seconds. Identify the unit-vector head.

1. A half-turn is $180^\circ = \pi$ radians. Multiply 30° by $\pi/180^\circ$.
2. $30\pi/180 = \pi/6$. Cancel the factor 30 before dividing.
3. For minutes, multiply by 60: $30 \times 60 = 1800'$.
4. For seconds, multiply by 3600: $30 \times 3600 = 108000''$. The 30° triangle values are derived in Example 6.

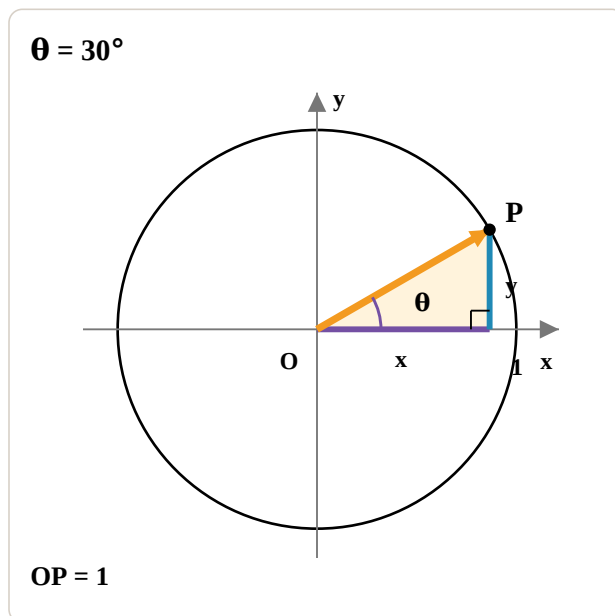


Figure 02. The orange arrow is the unit vector OP ; P is its head. Purple and blue legs show signed x and y projections.

DEGREES	RADIANS	TOTAL MINUTES	TOTAL SECONDS
30°	$\pi/6$	$1800'$	$108000''$
$\sin \theta = y$	$\cos \theta = x$	$\tan \theta = y/x$	
$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$	

HEAD COORDINATES: $P = (\sqrt{3}/2, 1/2)$
 UNIT VECTOR: $u = OP = (\sqrt{3}/2, 1/2)$; $\|u\| = 1$.

INVERSE CHECK. $\arcsin(1/2) = \arccos(\sqrt{3}/2) = \arctan(\sqrt{3}/3) = \pi/6 = 30^\circ$.

KEY IDEA. The four measures describe the same rotation. Angular minutes and seconds measure angle; they are not elapsed time.

VERIFY BY HAND. $1800' \div 60 = 30^\circ$; $108000'' \div 3600 = 30^\circ$. Also, $3/4 + 1/4 = 1$.

Example 03 | Degrees, minutes, and seconds together

Convert $22^\circ 30' 00''$ into decimal degrees, radians, and total arc units. Keep the angle exact.

1. Use $D + M/60 + S/3600$. Thus $22 + 30/60 + 0/3600 = 22 + 1/2 = 45/2$ degrees.

2. Multiply by $\pi/180$: $(45/2)(\pi/180) = \pi/8$.

3. Total minutes: $22 \times 60 + 30 = 1350'$. Total seconds: $1350 \times 60 = 81000''$.

4. Since 22.5° is half of 45° , its exact coordinates come from the half-angle rule proved and used in Example 22.

$$\theta = 22.5^\circ$$

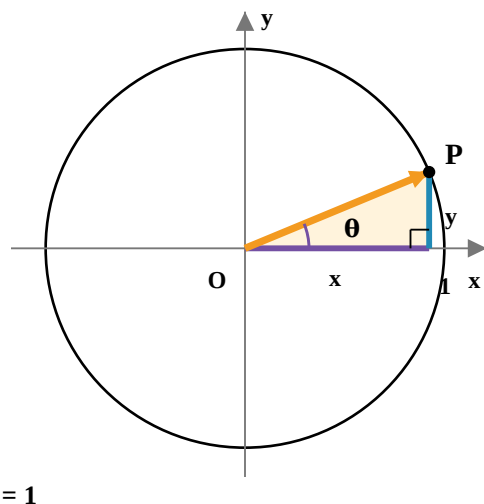


Figure 03. The orange arrow is the unit vector OP ; P is its head. Purple and blue legs show signed x and y projections.

DEGREES	RADIANS	TOTAL MINUTES	TOTAL SECONDS
22.5°	$\pi/8$	$1350'$	$81000''$

$\sin \theta = y$	$\cos \theta = x$	$\tan \theta = y/x$
$\sqrt{(2-\sqrt{2})}/2$	$\sqrt{(2+\sqrt{2})}/2$	$\sqrt{2}-1$

HEAD COORDINATES: $P = (\sqrt{(2+\sqrt{2})}/2, \sqrt{(2-\sqrt{2})}/2)$

UNIT VECTOR: $u = OP = (\sqrt{(2+\sqrt{2})}/2, \sqrt{(2-\sqrt{2})}/2)$; $\|u\| = 1$.

INVERSE CHECK. $\arcsin[\sqrt{(2-\sqrt{2})}/2] = \arccos[\sqrt{(2+\sqrt{2})}/2] = \arctan(\sqrt{2}-1) = \pi/8$.

KEY IDEA. $22^\circ 30'$ means 22.5° , not 22.30° . Base 60 subdivisions must be converted before decimal arithmetic.

VERIFY BY HAND. $[(2-\sqrt{2}) + (2+\sqrt{2})]/4 = 1$. The squared coordinates add to one.

Example 04 | Radians are arc length divided by radius

A circle of radius 6 has a counterclockwise arc of length 2π . Find θ and then its unit-circle point.

1. Use $\theta = s/r$ with θ in radians. Here $\theta = 2\pi/6 = \pi/3$.
2. Convert to degrees: $(\pi/3)(180/\pi) = 60^\circ$. The factors π cancel.
3. The original radius is 6; scaling the circle to radius 1 preserves the angle.
4. On the unit circle, the corresponding arc length equals $\pi/3$. The coordinates use the 60° triangle of Example 7.

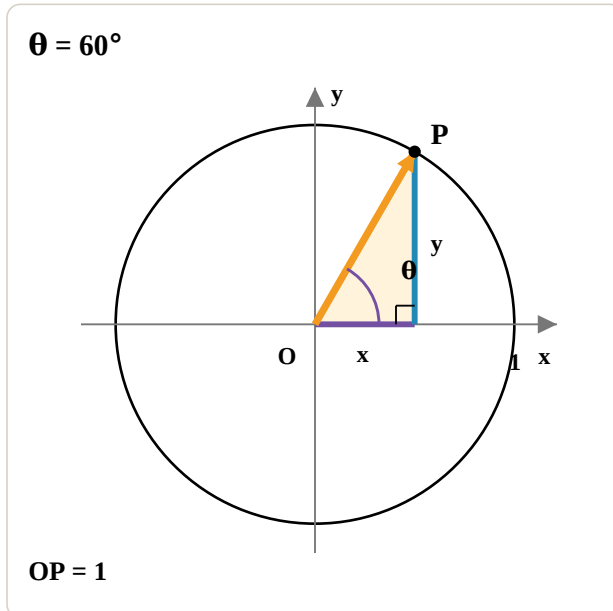


Figure 04. The orange arrow is the unit vector OP ; P is its head. Purple and blue legs show signed x and y projections.

DEGREES	RADIANS	TOTAL MINUTES	TOTAL SECONDS
60°	$\pi/3$	$3600'$	$216000''$
$\sin \theta = y$	$\cos \theta = x$	$\tan \theta = y/x$	
$\sqrt{3}/2$	$1/2$	$\sqrt{3}$	

HEAD COORDINATES: $P = (1/2, \sqrt{3}/2)$
 UNIT VECTOR: $\mathbf{u} = OP = (1/2, \sqrt{3}/2)$; $\|\mathbf{u}\| = 1$.

INVERSE CHECK. $\arcsin(\sqrt{3}/2) = \arccos(1/2) = \arctan(\sqrt{3}) = \pi/3 = 60^\circ$.

KEY IDEA. Arc length s and radius r must use the same length unit. The ratio s/r is then dimensionless.

VERIFY BY HAND. Recover the original arc: $r\theta = 6(\pi/3) = 2\pi$. An arc is curved; its length is not a triangle leg.

Chapter 2 | Build the exact angles from geometry

Two triangles supply the main values

A square produces the 45° - 45° - 90° triangle with side ratio $1:1:\sqrt{2}$. An equilateral triangle split by an altitude produces the 30° - 60° - 90° triangle with side ratio $1:\sqrt{3}:2$. These are geometric derivations, not numbers that must be looked up.

From a ratio to a coordinate

Dividing all sides by the hypotenuse makes the radius equal to 1. The adjacent leg becomes $x = \cos \theta$, and the opposite leg becomes $y = \sin \theta$. Once you know x and y , tangent follows by division whenever $x \neq 0$.

Axis points complete the first tools

At 0° , 90° , 180° , and 270° , one coordinate is zero. The unit-circle definition still works, even though a nondegenerate reference triangle does not exist. Tangent fails exactly where the x -coordinate is zero.

A compact memory pattern, with a reason

In order 0° , 30° , 45° , 60° , 90° , the sine values are $\sqrt{0}/2$, $\sqrt{1}/2$, $\sqrt{2}/2$, $\sqrt{3}/2$, $\sqrt{4}/2$. Cosine reads the same list backward. The geometric derivations justify the pattern; the pattern helps you rebuild the table quickly.

ANGLE	$\sin \theta$	$\cos \theta$	$\tan \theta$
0°	0	1	0
30°	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
45°	$\sqrt{2}/2$	$\sqrt{2}/2$	1
60°	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
90°	1	0	undefined

Use a straightedge and compass to construct an equilateral triangle or bisect an angle. A protractor can draw approximate angles; it does not prove an exact ratio. Radicals preserve exact lengths.

Build your own reference card: sketch the two special triangles and write their side ratios. Derive each function by division before consulting the table. Repeat the construction with a different hypotenuse length to see why scaling does not change the trigonometric ratios.

Example 05 | Derive the 45° triangle

Begin with a square of side 1. Draw its diagonal and derive the exact unit-circle coordinates.

1. The diagonal splits the square into two congruent right triangles. Each acute angle is 45°.
2. By Pythagoras, the diagonal is $\sqrt{1^2 + 1^2} = \sqrt{2}$.
3. Divide both legs by $\sqrt{2}$. Each normalized leg is $1/\sqrt{2}$.
4. Rationalize: $1/\sqrt{2} = \sqrt{2}/2$. The opposite and adjacent legs are equal, so $\tan 45^\circ = 1$.

$$\theta = 45^\circ$$

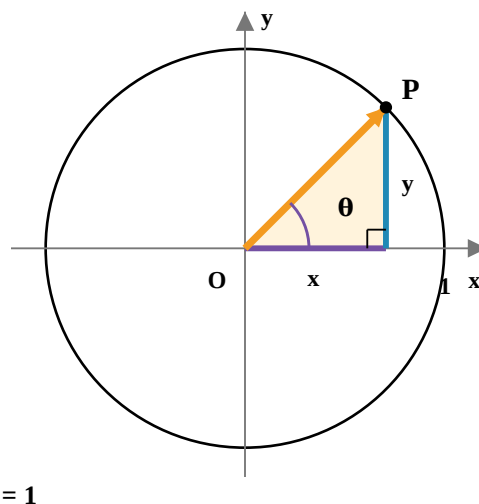


Figure 05. The orange arrow is the unit vector OP ; P is its head. Purple and blue legs show signed x and y projections.

DEGREES	RADIANS	TOTAL MINUTES	TOTAL SECONDS
45°	$\pi/4$	2700'	162000''

$\sin \theta = y$	$\cos \theta = x$	$\tan \theta = y/x$
$\sqrt{2}/2$	$\sqrt{2}/2$	1

HEAD COORDINATES: $P = (\sqrt{2}/2, \sqrt{2}/2)$
 UNIT VECTOR: $\mathbf{u} = OP = (\sqrt{2}/2, \sqrt{2}/2)$; $\|\mathbf{u}\| = 1$.

INVERSE CHECK. $\arcsin(\sqrt{2}/2) = \arccos(\sqrt{2}/2) = \arctan(1) = \pi/4 = 45^\circ$.

KEY IDEA. The radical $\sqrt{2}$ is an exact length. A decimal approximation is optional and is developed by hand in Example 25.

VERIFY BY HAND. $(\sqrt{2}/2)^2 + (\sqrt{2}/2)^2 = 1/2 + 1/2 = 1$.

Example 06 | Derive the 30° triangle

Split an equilateral triangle of side 2 down the middle. Use one half to find the 30° ratios.

1. Every angle of an equilateral triangle is 60°. Its altitude bisects one angle to make 30°.
2. The short leg is 1 and the hypotenuse is 2. The other leg is $\sqrt{4-1} = \sqrt{3}$.
3. At the 30° angle: opposite = 1, adjacent = $\sqrt{3}$, hypotenuse = 2.
4. Normalize by 2. Then $\tan \theta = (1/2)/(\sqrt{3}/2) = 1/\sqrt{3} = \sqrt{3}/3$.

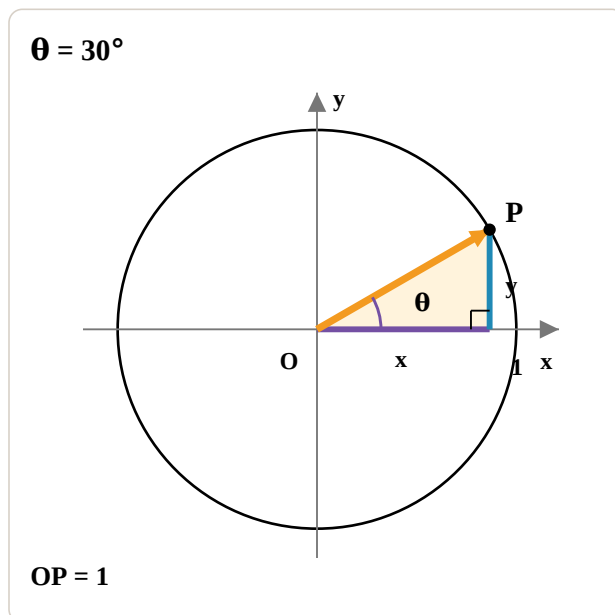


Figure 06. The orange arrow is the unit vector OP ; P is its head. Purple and blue legs show signed x and y projections.

DEGREES	RADIANS	TOTAL MINUTES	TOTAL SECONDS
30°	$\pi/6$	1800'	108000''

$\sin \theta = y$	$\cos \theta = x$	$\tan \theta = y/x$
1/2	$\sqrt{3}/2$	$\sqrt{3}/3$

HEAD COORDINATES: $P = (\sqrt{3}/2, 1/2)$
 UNIT VECTOR: $u = OP = (\sqrt{3}/2, 1/2)$; $\|u\| = 1$.

INVERSE CHECK. $\arcsin(1/2) = \arccos(\sqrt{3}/2) = \arctan(\sqrt{3}/3) = \pi/6$.

KEY IDEA. Label θ before naming the legs. The opposite and adjacent sides switch when you choose the other acute angle.

VERIFY BY HAND. The unit triangle satisfies $(\sqrt{3}/2)^2 + (1/2)^2 = 3/4 + 1/4 = 1$.

Example 07 | Use the other acute angle: 60°

Use the same 1, $\sqrt{3}$, 2 triangle, now with θ at the 60° corner.

1. The acute angles of a right triangle add to 90° . Thus the second angle is $90^\circ - 30^\circ = 60^\circ$.

2. Relative to 60° , the opposite side is $\sqrt{3}$ and the adjacent side is 1.

3. Divide by the hypotenuse 2: $\sin 60^\circ = \sqrt{3}/2$ and $\cos 60^\circ = 1/2$.

4. Their quotient is $\sqrt{3}$. Complementary angles exchange sine and cosine.

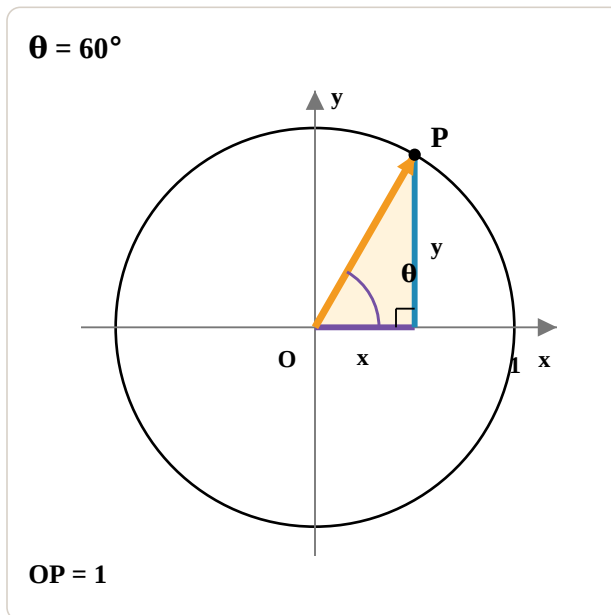


Figure 07. The orange arrow is the unit vector OP ; P is its head. Purple and blue legs show signed x and y projections.

DEGREES	RADIANS	TOTAL MINUTES	TOTAL SECONDS
60°	$\pi/3$	$3600'$	$216000''$

$\sin \theta = y$	$\cos \theta = x$	$\tan \theta = y/x$
$\sqrt{3}/2$	$1/2$	$\sqrt{3}$

HEAD COORDINATES: $P = (1/2, \sqrt{3}/2)$
 UNIT VECTOR: $\mathbf{u} = OP = (1/2, \sqrt{3}/2)$; $\|\mathbf{u}\| = 1$.

INVERSE CHECK. $\arcsin(\sqrt{3}/2) = \arccos(1/2) = \arctan(\sqrt{3}) = \pi/3$.

KEY IDEA. $\sin(90^\circ - \theta) = \cos \theta$ and $\cos(90^\circ - \theta) = \sin \theta$. These are cofunction identities.

VERIFY BY HAND. $\sin 60^\circ = \cos 30^\circ$; $\cos 60^\circ = \sin 30^\circ$. The longer leg lies opposite the larger acute angle.

Example 08 | An axis angle: 90°

Read the top of the circle. Explain why tangent has no value here.

1. A quarter-turn sends the unit vector from $(1,0)$ to $P = (0,1)$.
2. Therefore $\cos 90^\circ = 0$ and $\sin 90^\circ = 1$.
3. $\tan 90^\circ$ would require $y/x = 1/0$. Division by zero is undefined.
4. The reference triangle has collapsed to a line segment. Use coordinates at axis angles; do not pretend it is a nondegenerate triangle.

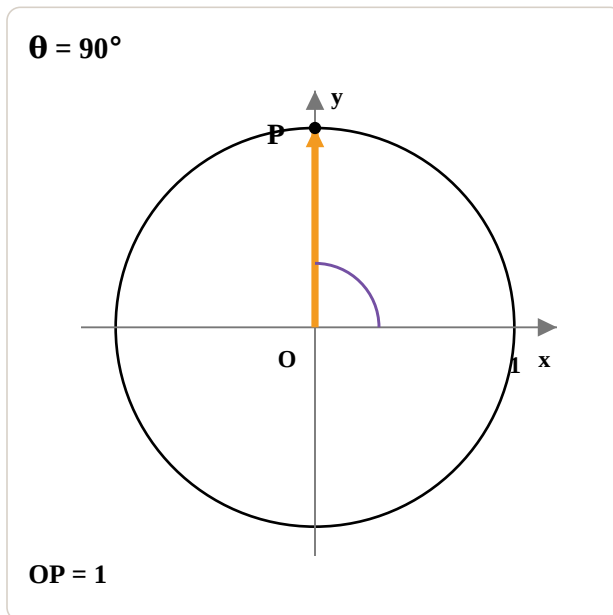


Figure 08. The orange arrow is the unit vector OP ; P is its head. At an axis angle, the reference triangle collapses.

DEGREES	RADIANS	TOTAL MINUTES	TOTAL SECONDS
90°	$\pi/2$	$5400'$	$324000''$

$\sin \theta = y$	$\cos \theta = x$	$\tan \theta = y/x$
1	0	undefined

HEAD COORDINATES: $P = (0, 1)$
 UNIT VECTOR: $u = OP = (0, 1)$; $\|u\| = 1$.

INVERSE CHECK. $\arcsin(1) = \arccos(0) = \pi/2$. There is no real number $\arctan(\text{undefined})$.

KEY IDEA. Undefined does not mean zero, and it does not mean an ordinary real number called infinity.

VERIFY BY HAND. The same rule makes tangent undefined at 270° . At 0° and 180° , tangent is 0 because $y = 0$ and $x \neq 0$.

Chapter 3 | Extend the triangle around the circle

Reference angle versus directed angle

The reference angle α is the acute angle between the terminal ray and the nearest part of the x-axis. The full directed angle θ can be negative or greater than 90° . For an obtuse or reflex θ , the right triangle contains α , while the rotation at O records θ .

Four quadrants, one set of magnitudes

Use $|x|$ and $|y|$ as triangle-leg lengths. Then restore the coordinate signs from the quadrant. Squaring the coordinates removes their signs, so $x^2 + y^2 = 1$ holds in every quadrant. Tangent inherits the sign of y/x .

Coterminal angles

Adding or subtracting $360^\circ k$, or $2\pi k$ radians, leaves the unit-vector head unchanged for any integer k . Reducing an angle makes calculation easier, but retain the original angle when recording how much rotation occurred.

Symmetry does most of the work

Reflecting across the x-axis changes (x,y) to $(x,-y)$. Reflecting across the y-axis changes (x,y) to $(-x,y)$. A half-turn changes (x,y) to $(-x,-y)$. These geometric facts generate the sign and periodicity identities.

QUADRANT	x	y	$\tan \theta$	REFERENCE
I: $0^\circ < \theta < 90^\circ$	+	+	+	$\alpha = \theta$
II: $90^\circ < \theta < 180^\circ$	−	+	−	$\alpha = 180^\circ - \theta$
III: $180^\circ < \theta < 270^\circ$	−	−	+	$\alpha = \theta - 180^\circ$
IV: $270^\circ < \theta < 360^\circ$	+	−	−	$\alpha = 360^\circ - \theta$

A reference angle is acute only away from the axes. Treat axis directions directly by coordinates. The formulas in the table assume θ has first been reduced to one positive revolution.

Draw a quadrant map before calculating. Mark the horizontal sign first and the vertical sign second. Read the ordered pair as $(\cos \theta, \sin \theta)$. This habit prevents the common error of reversing sine and cosine when copying a point from a diagram.

Example 09 | Quadrant II: 120°

Use a 60° reference angle to locate 120° and compute all three ratios.

- 120° lies between 90° and 180° , in quadrant II.
- The reference angle is $\alpha = 180^\circ - 120^\circ = 60^\circ$.
- Use the 60° magnitudes: $|x| = 1/2$ and $|y| = \sqrt{3}/2$. In quadrant II, x is negative and y is positive.
- Thus $\tan \theta = (\sqrt{3}/2)/(-1/2) = -\sqrt{3}$. The reference triangle contains α ; the full directed angle is θ .

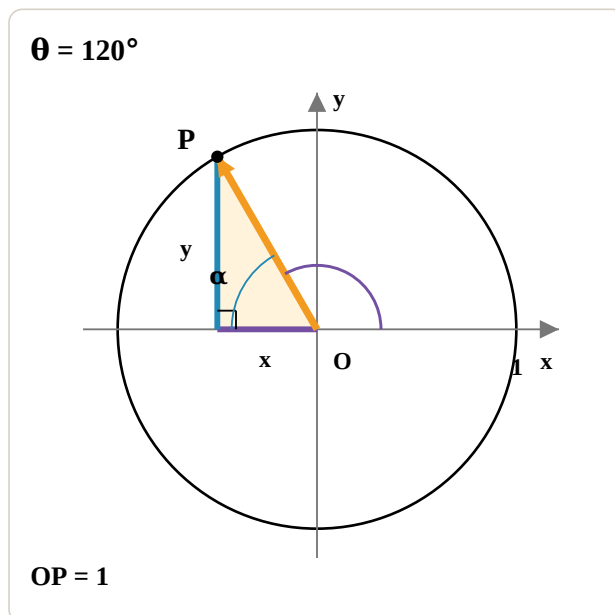


Figure 09. The orange arrow is the unit vector OP ; P is its head. Purple and blue legs show signed x and y projections.

DEGREES	RADIANS	TOTAL MINUTES	TOTAL SECONDS
120°	$2\pi/3$	$7200'$	$432000''$

$\sin \theta = y$	$\cos \theta = x$	$\tan \theta = y/x$
$\sqrt{3}/2$	$-1/2$	$-\sqrt{3}$

HEAD COORDINATES: $P = (-1/2, \sqrt{3}/2)$

UNIT VECTOR: $\mathbf{u} = OP = (-1/2, \sqrt{3}/2)$; $\|\mathbf{u}\| = 1$.

INVERSE CHECK. $\arcsin(\sin 120^\circ) = 60^\circ$; $\arccos(\cos 120^\circ) = 120^\circ$; $\arctan(\tan 120^\circ) = -60^\circ$.

KEY IDEA. A triangle has positive side lengths. Signed x and y coordinates carry quadrant information.

VERIFY BY HAND. $(-1/2)^2 + (\sqrt{3}/2)^2 = 1$. Sine is positive here; cosine and tangent are negative.

Example 10 | Quadrant II: 135°

Reflect the 45° point across the y-axis to find the point at 135° .

1. The reference angle is $180^\circ - 135^\circ = 45^\circ$.
2. A y-axis reflection changes x to $-x$ and keeps y unchanged.
3. Start from $(\sqrt{2}/2, \sqrt{2}/2)$; the reflected point is $(-\sqrt{2}/2, \sqrt{2}/2)$.
4. Divide y by x . Equal magnitudes with opposite signs give $\tan 135^\circ = -1$.

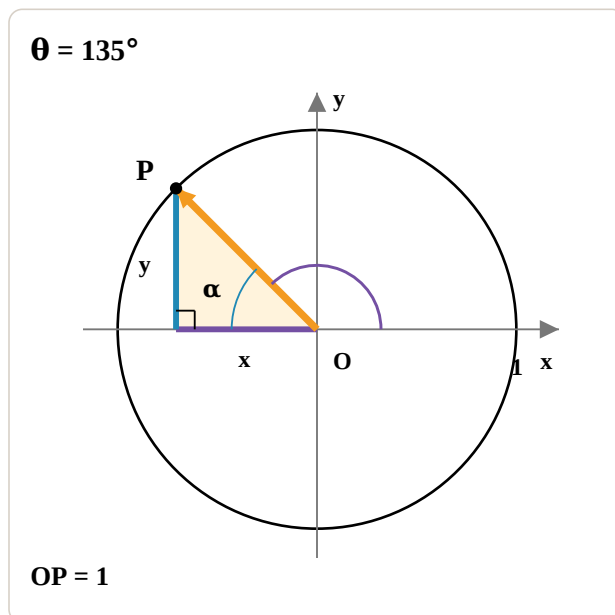


Figure 10. The orange arrow is the unit vector OP ; P is its head. Purple and blue legs show signed x and y projections.

DEGREES	RADIANS	TOTAL MINUTES	TOTAL SECONDS
135°	$3\pi/4$	$8100'$	$486000''$

$\sin \theta = y$	$\cos \theta = x$	$\tan \theta = y/x$
$\sqrt{2}/2$	$-\sqrt{2}/2$	-1

HEAD COORDINATES: $P = (-\sqrt{2}/2, \sqrt{2}/2)$
 UNIT VECTOR: $\mathbf{u} = OP = (-\sqrt{2}/2, \sqrt{2}/2)$; $\|\mathbf{u}\| = 1$.

INVERSE CHECK. $\arcsin(\sqrt{2}/2) = 45^\circ$; $\arccos(-\sqrt{2}/2) = 135^\circ$; $\arctan(-1) = -45^\circ$.

KEY IDEA. The principal inverse functions return angles in their own specified ranges. They need not return the original θ .

VERIFY BY HAND. $\cos(180^\circ - 45^\circ) = -\cos 45^\circ$ and $\sin(180^\circ - 45^\circ) = \sin 45^\circ$.

Example 11 | Quadrant III: 210°

Add a half-turn to the 30° vector and track the signs.

1. Write $210^\circ = 180^\circ + 30^\circ$. The reference angle is 30° .
2. A half-turn reverses both coordinates: (x,y) becomes $(-x,-y)$.
3. The point is $(-\sqrt{3}/2, -1/2)$. Both sine and cosine are negative.
4. The quotient of two negative numbers is positive:
 $\tan 210^\circ = (-1/2)/(-\sqrt{3}/2) = \sqrt{3}/3$.

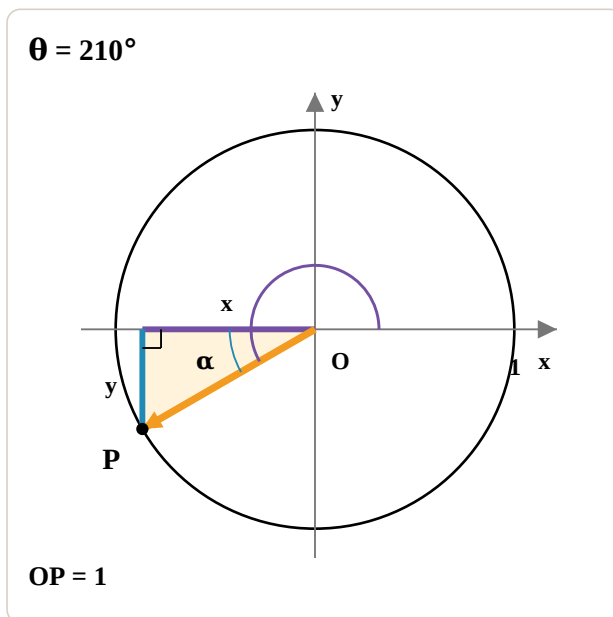


Figure 11. The orange arrow is the unit vector OP ; P is its head. Purple and blue legs show signed x and y projections.

DEGREES	RADIANS	TOTAL MINUTES	TOTAL SECONDS
210°	$7\pi/6$	$12600'$	$756000''$

$\sin \theta = y$	$\cos \theta = x$	$\tan \theta = y/x$
$-1/2$	$-\sqrt{3}/2$	$\sqrt{3}/3$

HEAD COORDINATES: $P = (-\sqrt{3}/2, -1/2)$

UNIT VECTOR: $\mathbf{u} = OP = (-\sqrt{3}/2, -1/2)$; $\|\mathbf{u}\| = 1$.

INVERSE CHECK. $\arcsin(-1/2) = -30^\circ$; $\arccos(-\sqrt{3}/2) = 150^\circ$; $\arctan(\sqrt{3}/3) = 30^\circ$.

KEY IDEA. Tangent has period 180° . Its sign alone cannot distinguish quadrant I from quadrant III.

VERIFY BY HAND. $\sin(\theta+180^\circ) = -\sin \theta$, $\cos(\theta+180^\circ) = -\cos \theta$, and $\tan(\theta+180^\circ) = \tan \theta$.

Example 12 | Quadrant IV: 300°

Use the 60° triangle below the x-axis.

1. The reference angle is $360^\circ - 300^\circ = 60^\circ$.
2. In quadrant IV, x is positive and y is negative.
3. Place the 60° magnitudes on the axes: $x = 1/2$ and $y = -\sqrt{3}/2$.
4. $\tan 300^\circ = (-\sqrt{3}/2)/(1/2) = -\sqrt{3}$. This is the same terminal ray as -60° .

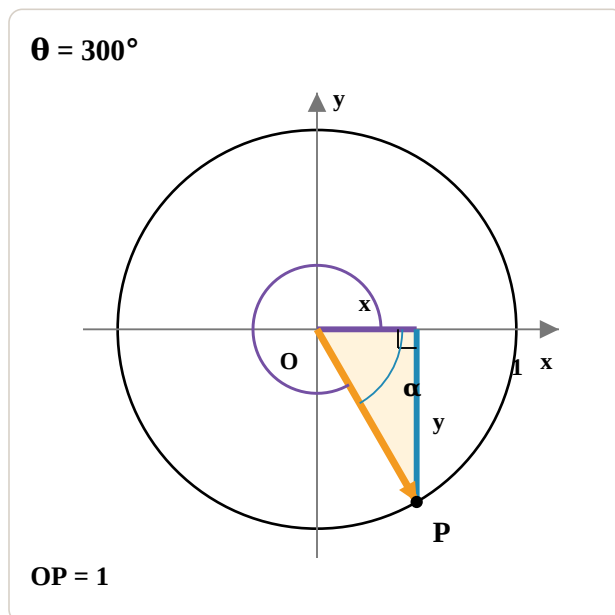


Figure 12. The orange arrow is the unit vector OP ; P is its head. Purple and blue legs show signed x and y projections.

DEGREES	RADIANS	TOTAL MINUTES	TOTAL SECONDS
300°	$5\pi/3$	$18000'$	$1080000''$

$\sin \theta = y$	$\cos \theta = x$	$\tan \theta = y/x$
$-\sqrt{3}/2$	$1/2$	$-\sqrt{3}$

HEAD COORDINATES: $P = (1/2, -\sqrt{3}/2)$

UNIT VECTOR: $\mathbf{u} = OP = (1/2, -\sqrt{3}/2)$; $\|\mathbf{u}\| = 1$.

INVERSE CHECK. $\arcsin(-\sqrt{3}/2) = -60^\circ$; $\arccos(1/2) = 60^\circ$; $\arctan(-\sqrt{3}) = -60^\circ$.

KEY IDEA. Always identify the quadrant before assigning signs to special-angle magnitudes.

VERIFY BY HAND. $\cos 300^\circ = \cos 60^\circ$ and $\sin 300^\circ = -\sin 60^\circ$. The reflection is across the x -axis.

Example 13 | Clockwise rotation: -30°

Find the values for a clockwise turn of 30° and compare them with 330° .

1. Negative angles turn clockwise from the positive x-axis.
2. Add a full turn: $-30^\circ + 360^\circ = 330^\circ$. The two angles have the same unit-vector head.
3. Reflection across the x-axis keeps cosine unchanged and reverses sine.
4. So $\sin(-30^\circ) = -1/2$, $\cos(-30^\circ) = \sqrt{3}/2$, and $\tan(-30^\circ) = -\sqrt{3}/3$.

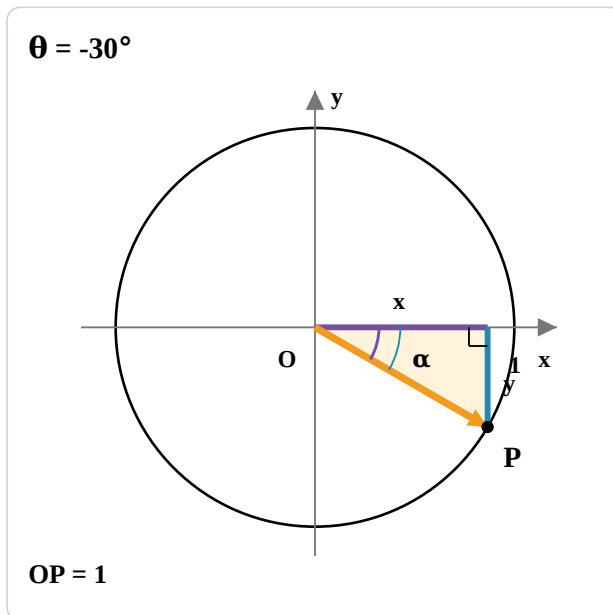


Figure 13. The orange arrow is the unit vector OP ; P is its head. Purple and blue legs show signed x and y projections.

DEGREES	RADIANS	TOTAL MINUTES	TOTAL SECONDS
-30°	$-\pi/6$	$-1800'$	$-108000''$

$\sin \theta = y$	$\cos \theta = x$	$\tan \theta = y/x$
$-1/2$	$\sqrt{3}/2$	$-\sqrt{3}/3$

HEAD COORDINATES: $P = (\sqrt{3}/2, -1/2)$

UNIT VECTOR: $\mathbf{u} = OP = (\sqrt{3}/2, -1/2)$; $\|\mathbf{u}\| = 1$.

INVERSE CHECK. $\arcsin(-1/2) = -30^\circ$; $\arccos(\sqrt{3}/2) = 30^\circ$; $\arctan(-\sqrt{3}/3) = -30^\circ$.

KEY IDEA. The signed totals $-1800'$ and $-108000''$ describe clockwise angular measure. They are not positive elapsed times.

VERIFY BY HAND. Sine and tangent are odd: $f(-\theta) = -f(\theta)$. Cosine is even: $\cos(-\theta) = \cos \theta$.

Example 14 | More than one turn: 390°

Reduce 390° to a coterminal angle and keep the original angular measure in the record.

1. Subtract one complete revolution: $390^\circ - 360^\circ = 30^\circ$.
2. The unit vector arrives at the same point as for 30° . Full turns do not change its final coordinates.
3. Convert the original angle: $390\pi/180 = 13\pi/6$ radians.
4. Use the 30° ratios. The diagram shows the terminal ray; 390° includes one extra complete turn.

$\theta = 390^\circ$

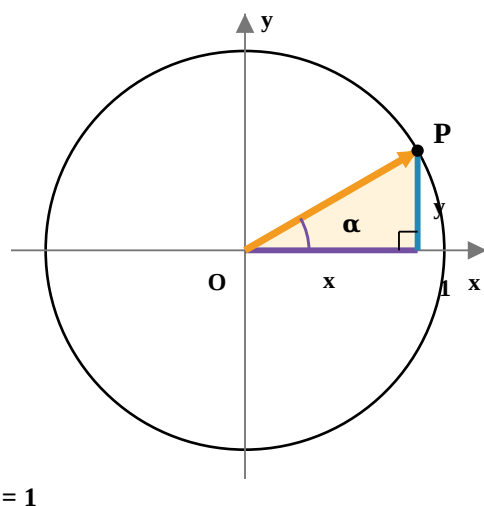


Figure 14. The orange arrow is the unit vector OP ; P is its head. Purple and blue legs show signed x and y projections.

DEGREES	RADIANS	TOTAL MINUTES	TOTAL SECONDS
390°	$13\pi/6$	$23400'$	$1404000''$

$\sin \theta = y$	$\cos \theta = x$	$\tan \theta = y/x$
$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$

HEAD COORDINATES: $P = (\sqrt{3}/2, 1/2)$
 UNIT VECTOR: $\mathbf{u} = OP = (\sqrt{3}/2, 1/2)$; $\|\mathbf{u}\| = 1$.

INVERSE CHECK. $\arcsin(1/2) = \arccos(\sqrt{3}/2) = \arctan(\sqrt{3}/3) = 30^\circ$, not 390° .

KEY IDEA. Coterminal angles share a head position, but they can record different amounts of rotation.

VERIFY BY HAND. $13\pi/6 - 2\pi = \pi/6$. The inverse functions cannot recover the number of completed turns from one coordinate.

Chapter 4 | Inverse functions recover selected angles

Why a principal range is necessary

Many angles have the same sine, cosine, or tangent. A function must give one output per input. Each inverse therefore uses a restricted range of angles. The chosen range is part of the definition, not an optional convention to ignore.

Inverse is not reciprocal

$\arcsin x$, $\arccos x$, and $\arctan x$ output angles. By contrast, $\csc \theta = 1/\sin \theta$, $\sec \theta = 1/\cos \theta$, and $\cot \theta = 1/\tan \theta$ where those expressions are defined. The notation $\sin^{-1}x$ usually means $\arcsin x$; it does not mean $1/\sin x$.

A pencil-and-paper inverse method

Check the input domain, match an exact ratio if possible, draw the permitted semicircle or half-plane, and choose the angle in the principal range. For a non-special value, an inverse expression may be the exact final answer. A controlled approximation can then be added if needed.

Recovering a direction from coordinates

The quotient y/x loses information about which opposite ray you started on. Determine the quadrant using both signs. A full direction can be selected in $[0^\circ, 360^\circ)$ or another stated interval, even when $\arctan(y/x)$ returns a different principal angle.

FUNCTION	REAL INPUT	RADIAN OUTPUT	DEGREE OUTPUT
$\arcsin x$	$-1 \leq x \leq 1$	$[-\pi/2, \pi/2]$	$[-90^\circ, 90^\circ]$
$\arccos x$	$-1 \leq x \leq 1$	$[0, \pi]$	$[0^\circ, 180^\circ]$
$\arctan x$	all real x	$(-\pi/2, \pi/2)$	$(-90^\circ, 90^\circ)$

Inverse composition needs care: $\sin(\arcsin x) = x$ on the real domain, but $\arcsin(\sin \theta)$ equals θ only when θ is in the principal arcsine range. The same kind of restriction applies to cosine and tangent.

Range check: an arcsine answer must be on the right semicircle, an arccosine answer on the upper semicircle, and an arctangent answer strictly between the two vertical directions on the right. Use these pictures to test whether a proposed answer is possible.

Example 15 | Inverse sine: choose the principal angle

Find $\arcsin(1/2)$ by reconstructing the unit-circle triangle.

1. $\arcsin$ asks for an angle whose sine is the given number.
2. Set $y = 1/2$. The principal arcsine range is -90° through 90° .
3. In that range, $x \geq 0$. Thus $x = \sqrt{1 - 1/4} = \sqrt{3}/2$.
4. Recognize the 30° triangle. The output is $\pi/6$ radians, or 30° .

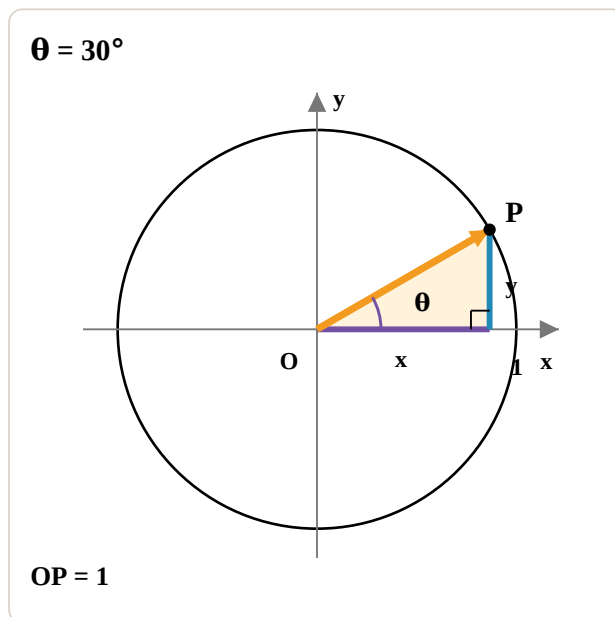


Figure 15. The orange arrow is the unit vector OP ; P is its head. Purple and blue legs show signed x and y projections.

DEGREES	RADIANS	TOTAL MINUTES	TOTAL SECONDS
30°	$\pi/6$	$1800'$	$108000''$

$\sin \theta = y$	$\cos \theta = x$	$\tan \theta = y/x$
$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$

HEAD COORDINATES: $P = (\sqrt{3}/2, 1/2)$
 UNIT VECTOR: $u = OP = (\sqrt{3}/2, 1/2)$; $\|u\| = 1$.

INVERSE CHECK. $\arcsin(1/2) = \pi/6$. The equation $\sin \theta = 1/2$ has additional solutions; see Example 18.

KEY IDEA. $\arcsin(1/2)$ is an angle. $\csc \theta = 1/\sin \theta$ is a reciprocal ratio. These are different operations.

VERIFY BY HAND. Substitute the result: $\sin(\pi/6) = 1/2$. The output $\pi/6$ lies inside $[-\pi/2, \pi/2]$.

Example 16 | Inverse cosine: a negative input

Find $\arccos(-\sqrt{2}/2)$ without a calculator.

1. Set $x = -\sqrt{2}/2$. The principal arccosine range is 0° through 180° .
2. A negative x -coordinate places the principal point in quadrant II.
3. Compute $y = +\sqrt{1-x^2} = \sqrt{1-1/2} = \sqrt{2}/2$.
4. The reference angle is 45° . Therefore $\theta = 180^\circ - 45^\circ = 135^\circ = 3\pi/4$.

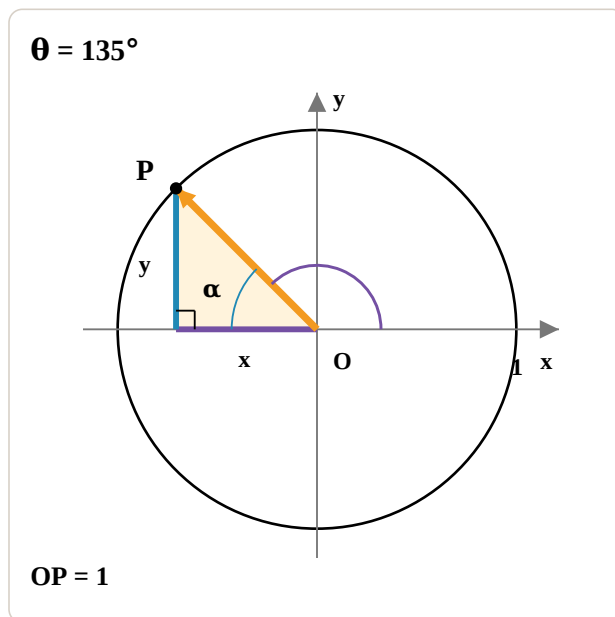


Figure 16. The orange arrow is the unit vector OP ; P is its head. Purple and blue legs show signed x and y projections.

DEGREES	RADIANS	TOTAL MINUTES	TOTAL SECONDS
135°	$3\pi/4$	$8100'$	$486000''$

$\sin \theta = y$	$\cos \theta = x$	$\tan \theta = y/x$
$\sqrt{2}/2$	$-\sqrt{2}/2$	-1

HEAD COORDINATES: $P = (-\sqrt{2}/2, \sqrt{2}/2)$
 UNIT VECTOR: $\mathbf{u} = OP = (-\sqrt{2}/2, \sqrt{2}/2)$; $\|\mathbf{u}\| = 1$.

INVERSE CHECK. $\arccos(-\sqrt{2}/2) = 3\pi/4$. Although 225° has the same cosine, it is outside the principal range.

KEY IDEA. For $\arccos$ inputs in $[-1, 1]$, choose the point on the upper semicircle, including its endpoints.

VERIFY BY HAND. $\cos(3\pi/4) = -\sqrt{2}/2$. Both the sign and the range condition are satisfied.

Example 17 | Inverse tangent: a negative ratio

Find $\arctan(-\sqrt{3})$ and recover the unit-vector coordinates.

1. The principal arctangent range is strictly between -90° and 90° .
2. The familiar positive angle with tangent $\sqrt{3}$ is 60° .
3. A negative tangent in the principal range selects -60° , in quadrant IV.
4. Use $x = 1/\sqrt{1+t^2}$ and $y = t/\sqrt{1+t^2}$ with $t = -\sqrt{3}$: $x = 1/2$, $y = -\sqrt{3}/2$.

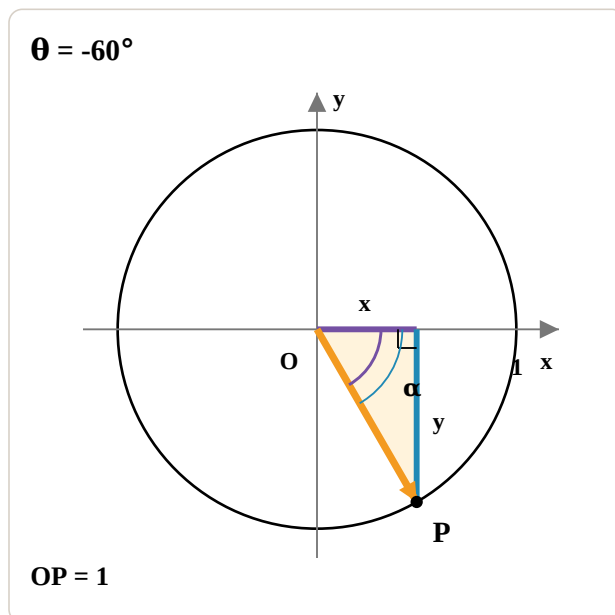


Figure 17. The orange arrow is the unit vector OP ; P is its head. Purple and blue legs show signed x and y projections.

DEGREES	RADIANS	TOTAL MINUTES	TOTAL SECONDS
-60°	$-\pi/3$	$-3600'$	$-216000''$

$\sin \theta = y$	$\cos \theta = x$	$\tan \theta = y/x$
$-\sqrt{3}/2$	$1/2$	$-\sqrt{3}$

HEAD COORDINATES: $P = (1/2, -\sqrt{3}/2)$

UNIT VECTOR: $\mathbf{u} = OP = (1/2, -\sqrt{3}/2)$; $\|\mathbf{u}\| = 1$.

INVERSE CHECK. $\arctan(-\sqrt{3}) = -\pi/3$. The angle 120° has the same tangent but is not the principal output.

KEY IDEA. The endpoints $\pm 90^\circ$ are excluded because tangent is undefined there. Every finite real input has a principal arctangent.

VERIFY BY HAND. $y/x = (-\sqrt{3}/2)/(1/2) = -\sqrt{3}$, and x is positive as required in the principal tangent range.

Example 18 | An inverse value versus all solutions

Solve $\sin \theta = 1/2$ for $0^\circ \leq \theta < 360^\circ$, then describe every real solution.

1. The horizontal line $y = 1/2$ meets the unit circle twice.
2. The first point P is at 30° . Reflect it across the y-axis to obtain Q at 150° .
3. In one revolution, $\theta = 30^\circ$ or 150° . The inverse value $\arcsin(1/2)$ gives only 30° .
4. All solutions: $\theta = 30^\circ + 360^\circ k$ or $150^\circ + 360^\circ k$, where k is any integer. In radians: $\pi/6 + 2\pi k$ or $5\pi/6 + 2\pi k$.

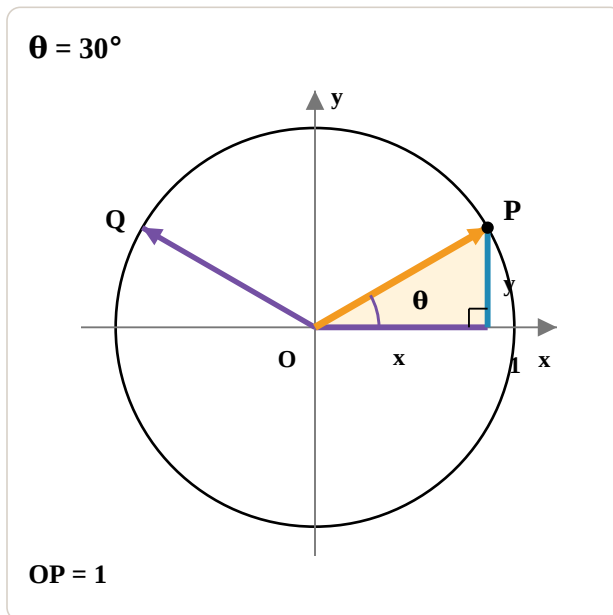


Figure 18. The orange arrow is the unit vector $\overrightarrow{OP}$; P is its head. The purple arrow ends at Q.

DEGREES	RADIANS	TOTAL MINUTES	TOTAL SECONDS
30°	$\pi/6$	$1800'$	$108000''$

$\sin \theta = y$	$\cos \theta = x$	$\tan \theta = y/x$
$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$

HEAD COORDINATES: $P = (\sqrt{3}/2, 1/2)$
 UNIT VECTOR: $\mathbf{u} = \overrightarrow{OP} = (\sqrt{3}/2, 1/2)$; $\|\mathbf{u}\| = 1$.

INVERSE CHECK. Principal inverse: $\arcsin(1/2) = 30^\circ$. Second point: $Q = (-\sqrt{3}/2, 1/2)$, with $\tan 150^\circ = -\sqrt{3}/3$.

KEY IDEA. For the second solution, $150^\circ = 5\pi/6 = 9000' = 540000''$. The record below selects P at 30° .

VERIFY BY HAND. Both points have the same height. The two x-values have opposite signs, so their cosines differ.

Chapter 5 | Alternative forms that create new values

Cartesian and parametric forms

The equation $x^2 + y^2 = 1$ describes the circle without choosing an angle. Its trigonometric parametrization is $x = \cos \theta$, $y = \sin \theta$. The vector form $u = (\cos \theta, \sin \theta)$ adds direction; the polar form is $(r, \theta) = (1, \theta)$. These are descriptions of the same points.

The slope form

If θ is in the principal tangent range and $t = \tan \theta$, then $x = 1/\sqrt{1+t^2}$ and $y = t/\sqrt{1+t^2}$. The choice $x > 0$ selects the correct ray. The opposite point has the same slope, which is why the quadrant must still be known.

The rational half-angle parameter

Put $t = \tan(\theta/2)$. The identities $\cos \theta = (1-t^2)/(1+t^2)$ and $\sin \theta = 2t/(1+t^2)$ turn rational t into rational unit coordinates. Algebra checks that the squares add to 1. The point $(-1, 0)$ appears as a limit as $|t|$ grows without bound.

Angles can be built from other angles

Addition, subtraction, half-angle, and complementary-angle identities extend the 30° and 45° toolkit. Use the quadrant to select signs when a square root appears. Normalization also supplies exact ratios from any known right triangle.

TOOL	LEFT SIDE	RIGHT SIDE
Sum	$\sin(A+B)$	$\sin A \cos B + \cos A \sin B$
Difference	$\cos(A-B)$	$\cos A \cos B + \sin A \sin B$
Half-angle	$\sin^2(A/2)$	$(1-\cos A)/2$
Half-angle	$\cos^2(A/2)$	$(1+\cos A)/2$

The tangent-line interpretation is another form: extend the ray to the vertical line $x = 1$. Its intersection is $(1, \tan \theta)$, whenever $\cos \theta \neq 0$. Tangent is a slope and a signed vertical distance on that line.

Derivation of the rational form: write the double-angle formulas over the denominator $\sin^2(\theta/2) + \cos^2(\theta/2) = 1$. Divide numerator and denominator by $\cos^2(\theta/2)$. Both coordinates then have denominator $1+t^2$, with $t = \tan(\theta/2)$. This explains why t represents the tangent of half the angle.

Example 19 | A rational parameter for the circle

Use $t = \tan(\theta/2) = 1/2$ to generate an exact point without first knowing the angle in degrees.

1. Let $t = 1/2$. Use $x = (1-t^2)/(1+t^2)$ and $y = 2t/(1+t^2)$.
2. Compute $1-t^2 = 3/4$ and $1+t^2 = 5/4$.
3. Then $x = (3/4)/(5/4) = 3/5$ and $y = 1/(5/4) = 4/5$.
4. $\tan \theta = y/x = 4/3$. The angle is $\beta = 2 \arctan(1/2)$, an exact expression.

$$\theta = \beta = 2 \arctan(1/2)$$

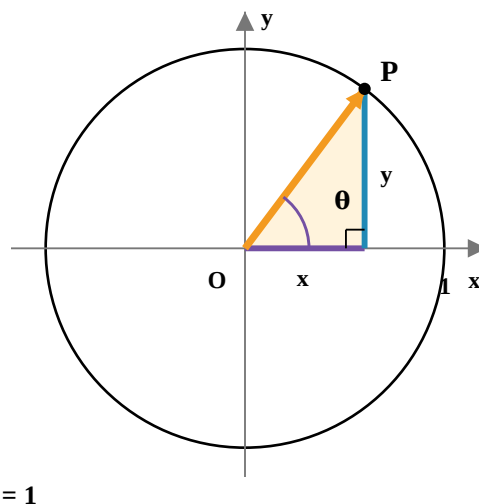


Figure 19. The orange arrow is the unit vector OP ; P is its head. Purple and blue legs show signed x and y projections.

DEGREES	RADIANS	TOTAL MINUTES	TOTAL SECONDS
$180\beta/\pi$ degrees	β radians	$10800\beta/\pi$ arcmin	$648000\beta/\pi$ arcsec

$\sin \theta = y$	$\cos \theta = x$	$\tan \theta = y/x$
$4/5$	$3/5$	$4/3$

HEAD COORDINATES: $P = (3/5, 4/5)$
 UNIT VECTOR: $\mathbf{u} = OP = (3/5, 4/5)$; $\|\mathbf{u}\| = 1$.

INVERSE CHECK. $\beta = 2 \arctan(1/2) = \arcsin(4/5) = \arccos(3/5) = \arctan(4/3)$, in quadrant I.

KEY IDEA. A rational t gives rational coordinates. This parameter covers the circle except $(-1,0)$ for finite real t .

VERIFY BY HAND. $(3/5)^2 + (4/5)^2 = 1$. Also $y/(1+x) = (4/5)/(8/5) = 1/2 = t$.

Example 20 | Subtract angles to obtain 15°

Derive $\sin 15^\circ$, $\cos 15^\circ$, and $\tan 15^\circ$ from $45^\circ - 30^\circ$.

1. $\sin(A-B) = \sin A \cos B - \cos A \sin B$.
2. $\sin 15^\circ = (\sqrt{2}/2)(\sqrt{3}/2) - (\sqrt{2}/2)(1/2) = (\sqrt{6}-\sqrt{2})/4$.
3. $\cos(A-B) = \cos A \cos B + \sin A \sin B$. Hence $\cos 15^\circ = (\sqrt{6}+\sqrt{2})/4$.
4. $\tan 15^\circ = (1-1/\sqrt{3})/(1+1/\sqrt{3}) = (\sqrt{3}-1)/(\sqrt{3}+1) = 2-\sqrt{3}$.

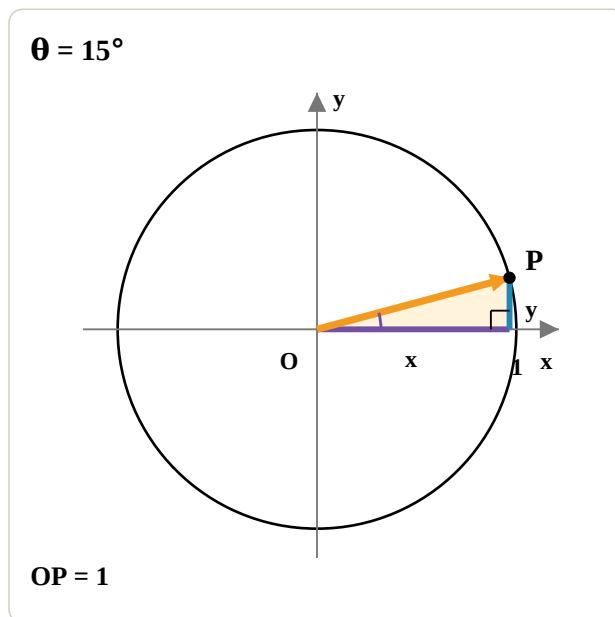


Figure 20. The orange arrow is the unit vector OP ; P is its head. Purple and blue legs show signed x and y projections.

DEGREES	RADIANS	TOTAL MINUTES	TOTAL SECONDS
15°	$\pi/12$	$900'$	$54000''$

$\sin \theta = y$	$\cos \theta = x$	$\tan \theta = y/x$
$(\sqrt{6}-\sqrt{2})/4$	$(\sqrt{6}+\sqrt{2})/4$	$2-\sqrt{3}$

HEAD COORDINATES: $P = ((\sqrt{6}+\sqrt{2})/4, (\sqrt{6}-\sqrt{2})/4)$
UNIT VECTOR: $u = OP = ((\sqrt{6}+\sqrt{2})/4, (\sqrt{6}-\sqrt{2})/4)$; $\|u\| = 1$.

INVERSE CHECK. The principal arcsin, arccos, and arctan of the listed ratios each return $\pi/12 = 15^\circ$.

KEY IDEA. Rationalize the tangent quotient by multiplying top and bottom by $\sqrt{3}-1$. The denominator becomes $3-1 = 2$.

VERIFY BY HAND. The angle lies in quadrant I, so all ratios are positive. The small y -coordinate agrees with a shallow angle.

Example 21 | Add angles to obtain 75°

Use $45^\circ + 30^\circ$ and compare the result with the 15° point.

1. $\sin(A+B) = \sin A \cos B + \cos A \sin B$.
2. $\sin 75^\circ = (\sqrt{2}/2)(\sqrt{3}/2) + (\sqrt{2}/2)(1/2) = (\sqrt{6}+\sqrt{2})/4$.
3. $\cos(A+B) = \cos A \cos B - \sin A \sin B$. Thus $\cos 75^\circ = (\sqrt{6}-\sqrt{2})/4$.
4. $\tan 75^\circ = (\sqrt{3}+1)/(\sqrt{3}-1) = 2+\sqrt{3}$. The coordinates of 15° have exchanged positions.

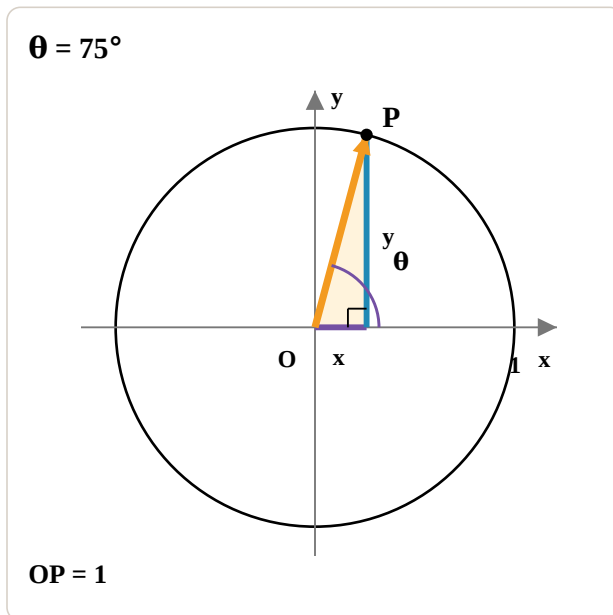


Figure 21. The orange arrow is the unit vector OP ; P is its head. Purple and blue legs show signed x and y projections.

DEGREES	RADIANS	TOTAL MINUTES	TOTAL SECONDS
75°	$5\pi/12$	$4500'$	$270000''$

$\sin \theta = y$	$\cos \theta = x$	$\tan \theta = y/x$
$(\sqrt{6}+\sqrt{2})/4$	$(\sqrt{6}-\sqrt{2})/4$	$2+\sqrt{3}$

HEAD COORDINATES: $P = ((\sqrt{6}-\sqrt{2})/4, (\sqrt{6}+\sqrt{2})/4)$

UNIT VECTOR: $u = OP = ((\sqrt{6}-\sqrt{2})/4, (\sqrt{6}+\sqrt{2})/4)$; $\|u\| = 1$.

INVERSE CHECK. The principal inverses of the listed ratios return $5\pi/12 = 75^\circ$.

KEY IDEA. Since $75^\circ + 15^\circ = 90^\circ$, the cofunction identities provide an independent check of the addition formulas.

VERIFY BY HAND. $(2-\sqrt{3})(2+\sqrt{3}) = 1$. Therefore $\tan 15^\circ$ and $\tan 75^\circ$ are reciprocals, as complementary acute angles require.

Example 22 | Halve 45° to obtain 22.5°

Derive the exact values used in the DMS example from the double-angle identities.

1. $\cos 2u = 1 - 2\sin^2 u = 2\cos^2 u - 1$. Solve each equation for the squared ratio.
2. $\sin^2 u = (1 - \cos 2u)/2$. With $u = 22.5^\circ$, $\cos 2u = \sqrt{2}/2$.
3. $\sin u = \sqrt{(2 - \sqrt{2})/2}$ and $\cos u = \sqrt{(2 + \sqrt{2})/2}$. Choose positive roots in quadrant I.
4. Use $\tan u = \sin 45^\circ / (1 + \cos 45^\circ) = \sqrt{2}/(2 + \sqrt{2}) = \sqrt{2} - 1$.

$\theta = 22.5^\circ$

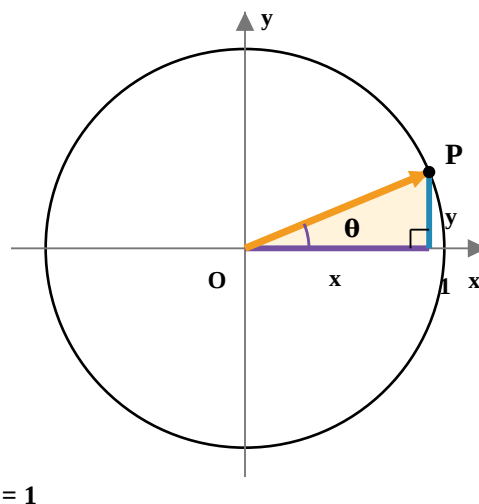


Figure 22. The orange arrow is the unit vector OP ; P is its head. Purple and blue legs show signed x and y projections.

DEGREES	RADIANS	TOTAL MINUTES	TOTAL SECONDS
22.5°	$\pi/8$	$1350'$	$81000''$

$\sin \theta = y$	$\cos \theta = x$	$\tan \theta = y/x$
$\sqrt{(2 - \sqrt{2})/2}$	$\sqrt{(2 + \sqrt{2})/2}$	$\sqrt{2} - 1$

HEAD COORDINATES: $P = (\sqrt{(2 + \sqrt{2})/2}, \sqrt{(2 - \sqrt{2})/2})$
 UNIT VECTOR: $u = OP = (\sqrt{(2 + \sqrt{2})/2}, \sqrt{(2 - \sqrt{2})/2})$; $\|u\| = 1$.

INVERSE CHECK. The principal inverses of these ratios return $\pi/8 = 22^\circ 30' 00''$.

KEY IDEA. A half-angle square root needs a sign choice. The quadrant of the half-angle determines that sign.

VERIFY BY HAND. $2\sin^2(22.5^\circ) = 1 - \sqrt{2}/2$, which reconstructs $\cos 45^\circ$ through the double-angle formula.

Example 23 | Use a complement to obtain 67.5°

Find the exact values at 67.5° without repeating the half-angle derivation.

1. Write $67.5^\circ = 90^\circ - 22.5^\circ$.
2. Exchange the sine and cosine of 22.5° : $\sin 67.5^\circ = \sqrt{(2+\sqrt{2})/2}$.
3. Similarly, $\cos 67.5^\circ = \sqrt{(2-\sqrt{2})/2}$.
4. $\tan 67.5^\circ = 1/(\sqrt{2}-1)$. Multiply by $(\sqrt{2}+1)/(\sqrt{2}+1)$ to obtain $\sqrt{2}+1$.

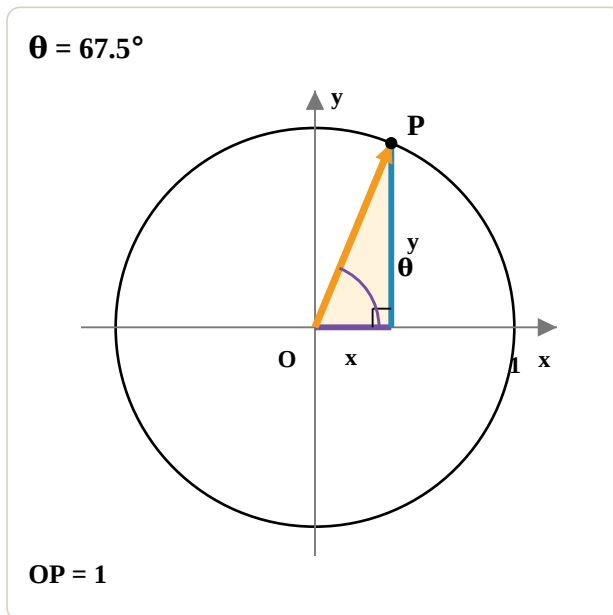


Figure 23. The orange arrow is the unit vector OP ; P is its head. Purple and blue legs show signed x and y projections.

DEGREES	RADIANS	TOTAL MINUTES	TOTAL SECONDS
67.5°	$3\pi/8$	$4050'$	$243000''$

$\sin \theta = y$	$\cos \theta = x$	$\tan \theta = y/x$
$\sqrt{(2+\sqrt{2})/2}$	$\sqrt{(2-\sqrt{2})/2}$	$\sqrt{2}+1$

HEAD COORDINATES: $P = (\sqrt{(2-\sqrt{2})/2}, \sqrt{(2+\sqrt{2})/2})$

UNIT VECTOR: $\mathbf{u} = OP = (\sqrt{(2-\sqrt{2})/2}, \sqrt{(2+\sqrt{2})/2})$; $\|\mathbf{u}\| = 1$.

INVERSE CHECK. The principal inverses of the listed ratios return $3\pi/8 = 67^\circ 30' 00''$.

KEY IDEA. A reflection across the line $y = x$ swaps the coordinates. That geometric action explains the complementary pair.

VERIFY BY HAND. $(\sqrt{2}-1)(\sqrt{2}+1) = 1$. The sine at 67.5° is greater than its cosine because the angle exceeds 45° .

Example 24 | Normalize a vector in quadrant II

A direction vector is $\mathbf{v} = (-5, 12)$. Find its unit vector, ratios, and a correct exact angle expression.

1. Find the length: $\|\mathbf{v}\| = \sqrt{(-5)^2 + 12^2} = \sqrt{169} = 13$.
2. Divide the components by 13: $\mathbf{u} = (-5/13, 12/13)$.
3. The reference angle is $\arctan(12/5)$. Since $x < 0$ and $y > 0$, θ lies in quadrant II.
4. Let $\varphi = \pi - \arctan(12/5)$. Its cosine is negative and sine positive. The tangent is $-12/5$.

$$\theta = \varphi = \pi - \arctan(12/5)$$

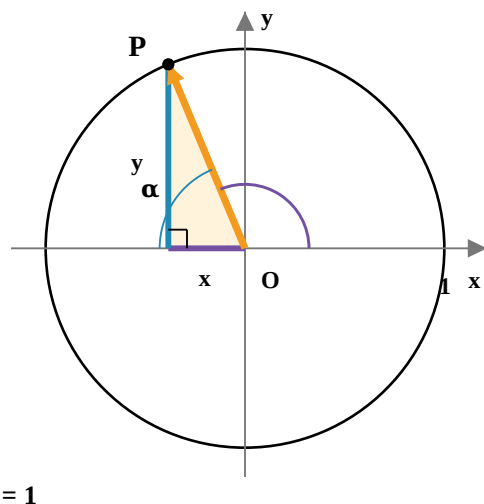


Figure 24. The orange arrow is the unit vector $\mathbf{OP}$; P is its head. Purple and blue legs show signed x and y projections.

DEGREES	RADIANS	TOTAL MINUTES	TOTAL SECONDS
$180\varphi/\pi$ degrees	φ radians	$10800\varphi/\pi$ arcmin	$648000\varphi/\pi$ arcsec

$\sin \theta = y$	$\cos \theta = x$	$\tan \theta = y/x$
$12/13$	$-5/13$	$-12/5$

HEAD COORDINATES: $P = (-5/13, 12/13)$

UNIT VECTOR: $\mathbf{u} = \mathbf{OP} = (-5/13, 12/13)$; $\|\mathbf{u}\| = 1$.

INVERSE CHECK. $\arccos(-5/13) = \varphi$. $\arcsin(12/13) = \pi - \varphi$; $\arctan(-12/5) = \varphi - \pi$.

KEY IDEA. Using $\arctan(y/x)$ alone would return a quadrant IV angle. Recover the quadrant from the signs of both coordinates.

VERIFY BY HAND. $25/169 + 144/169 = 1$. Multiplying $\mathbf{u}$ by 13 recovers $(-5, 12)$.

Chapter 6 | Calculate beyond the special angles

Exact form comes first

There is no short familiar radical formula for every angle. An exact expression such as $\sin(1/5)$ or $\arctan(1/10)$ is valid mathematics. When a decimal is useful, a hand method should also state why its error is small.

Build numbers through bounds

To approximate $\sqrt{a}$, find two positive numbers whose squares lie below and above a . Refine the interval with more decimal places. Newton averaging $g \rightarrow (g+a/g)/2$ converges rapidly from a positive guess; interval checks make the resulting accuracy visible.

Power series are algorithms on paper

For small radian x , powers shrink rapidly. Factorials in the denominators make the sine and cosine terms shrink even faster. The terms alternate in sign; when their magnitudes decrease to zero, the next omitted term bounds the truncation error. Reduce large angles before using this shortcut.

An article: what working by hand teaches

A calculator reports a value; a derivation explains its structure. Similar triangles show why a ratio depends on angle, symmetry predicts signs, and exact radicals preserve relationships that rounding hides. Paper computation also reveals the difference between measuring a drawing and proving a value. Efficient hand work combines geometry, algebra, a few well-chosen identities, and explicit error bounds.

FUNCTION	SERIES
$\sin x$	$x - x^3/3! + x^5/5! - x^7/7! + \dots$
$\cos x$	$1 - x^2/2! + x^4/4! - x^6/6! + \dots$
$\arctan z$	$z - z^3/3 + z^5/5 - z^7/7 + \dots \ (z \leq 1)$

Use radians in these series. $n! = 1 \times 2 \times \dots \times n$, with $0! = 1$. A practical π interval is $3.14159 < \pi < 3.14160$. It can be obtained by bounding the arctangent series in Machin's identity $\pi/4 = 4 \arctan(1/5) - \arctan(1/239)$.

Long division is enough to convert each final fraction into a decimal. Retain guard digits, bracket the omitted series terms, and round only after checking the interval. A decimal with many digits is useful only when the calculation justifies those digits.

Example 25 | Approximate a radical with pencil arithmetic

Keep the 45° values exact, then build a decimal interval for $\sqrt{2}/2$ by squaring simple decimals.

1. $1.4^2 = 1.96$ and $1.5^2 = 2.25$, so $1.4 < \sqrt{2} < 1.5$.
2. Refine: $1.41^2 = 1.9881$ and $1.42^2 = 2.0164$.
3. Refine again: $1.414^2 = 1.999396$ and $1.415^2 = 2.002225$.
4. Divide the final bounds by 2: $0.707 < \sqrt{2}/2 < 0.7075$.
The fraction with $\sqrt{2}$ remains the exact answer.

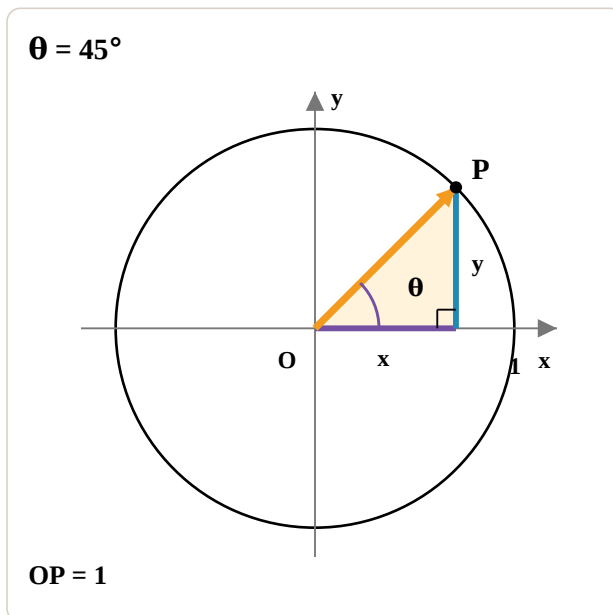


Figure 25. The orange arrow is the unit vector OP ; P is its head. Purple and blue legs show signed x and y projections.

DEGREES	RADIANS	TOTAL MINUTES	TOTAL SECONDS
45°	$\pi/4$	$2700'$	$162000''$
$\sin \theta = y$	$\cos \theta = x$	$\tan \theta = y/x$	
$\sqrt{2}/2$	$\sqrt{2}/2$	1	

HEAD COORDINATES: $P = (\sqrt{2}/2, \sqrt{2}/2)$
UNIT VECTOR: $\mathbf{u} = OP = (\sqrt{2}/2, \sqrt{2}/2)$; $\|\mathbf{u}\| = 1$.

INVERSE CHECK. $\arcsin(\sqrt{2}/2) = \arccos(\sqrt{2}/2) = \pi/4$. Rounding a coordinate does not preserve exact inverse identities.

KEY IDEA. A second method is Newton averaging: $g_{\text{new}} = (g + 2/g)/2$. Starting at $3/2$ gives $17/12$, then $577/408$.

VERIFY BY HAND. Square the bounds with integer multiplication, then place the decimal point. Every displayed bound can be checked by hand.

Example 26 | Find sine and cosine at 0.2 radians

For a non-special angle $\theta = 1/5$ radian, compute controlled approximations from power series.

- 1. Use radians. $\sin x = x - x^3/6 + x^5/120 - \dots$; $\cos x = 1 - x^2/2 + x^4/24 - \dots$.
- 2. At $x = 1/5$: $S = 1/5 - 1/750 + 1/375000 = 74501/375000 \approx 0.198669333$.
- 3. $C = 1 - 1/50 + 1/15000 = 14701/15000 \approx 0.980066667$.
- 4. The next omitted terms bound the errors: $|\sin x - S| < 0.000000003$ and $|\cos x - C| < 0.000000089$.

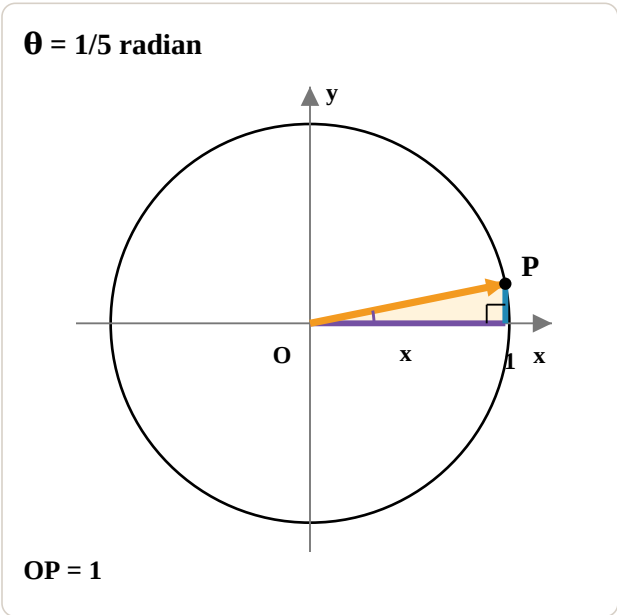


Figure 26. The orange arrow is the unit vector OP ; P is its head. Purple and blue legs show signed x and y projections.

DEGREES	RADIANS	TOTAL MINUTES	TOTAL SECONDS
$36/\pi$ degrees	$1/5$	$2160/\pi$ arcmin	$129600/\pi$ arcsec

$\sin \theta = y$	$\cos \theta = x$	$\tan \theta = y/x$
$\sin(1/5) \approx 0.198669$	$\cos(1/5) \approx 0.980067$	$\tan(1/5) \approx 0.202710$

HEAD COORDINATES: $P = (\cos(1/5), \sin(1/5))$
UNIT VECTOR: $u = OP = (\cos(1/5), \sin(1/5)); \|u\| = 1$.

INVERSE CHECK. All three principal inverse compositions return exactly $1/5$ radian when their exact ratios are used.

KEY IDEA. For these decreasing alternating terms, the true value lies between consecutive partial sums. Divide the sine interval by the positive cosine interval to bound tangent.

VERIFY BY HAND. The tangent lies between $(S - 1/393750000)/C$ and $S/(C - 1/11250000)$, giving $0.20271001 < \tan(1/5) < 0.20271004$.

Example 27 | Approximate an inverse angle by hand

Find a small angle with tangent $1/10$. First find exact coordinates; then approximate its angle.

1. Use a right triangle with adjacent 10 and opposite 1. Its hypotenuse is $\sqrt{101}$.

2. Let $\alpha = \arctan(1/10)$. Then $\cos \alpha = 10/\sqrt{101}$ and $\sin \alpha = 1/\sqrt{101}$ exactly.

3. For $|z| \leq 1$: $\arctan z = z - z^3/3 + z^5/5 - \dots$. With $z = 1/10$, use $T = 1/10 - 1/3000 + 1/500000$.

4. $T = 0.099668666\dots$ radians. The next term is $1/70000000$, so $T - 1/70000000 < \alpha < T$.

$$\theta = \arctan(1/10)$$

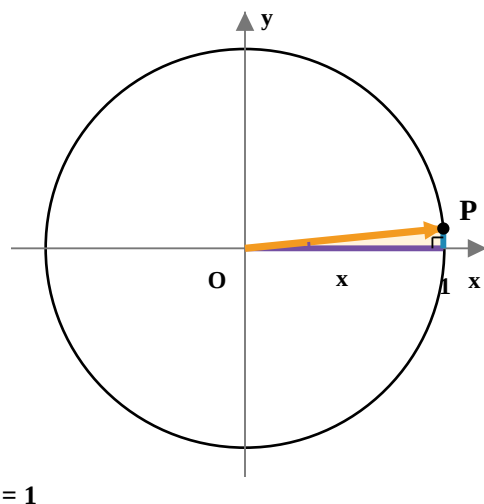


Figure 27. The orange arrow is the unit vector OP ; P is its head. Purple and blue legs show signed x and y projections.

DEGREES	RADIANS	TOTAL MINUTES	TOTAL SECONDS
$180\alpha/\pi$ degrees	$\alpha = \arctan(1/10)$	$10800\alpha/\pi$ arcmin	$648000\alpha/\pi$ arcsec

$\sin \theta = y$	$\cos \theta = x$	$\tan \theta = y/x$
$1/\sqrt{101}$	$10/\sqrt{101}$	$1/10$

HEAD COORDINATES: $P = (10/\sqrt{101}, 1/\sqrt{101})$

UNIT VECTOR: $u = OP = (10/\sqrt{101}, 1/\sqrt{101})$; $\|u\| = 1$.

INVERSE CHECK. $\alpha = \arctan(1/10) = \arcsin(1/\sqrt{101}) = \arccos(10/\sqrt{101})$. Numerically, $\alpha \approx 0.09966865$ rad.

KEY IDEA. Multiplying the radian interval by $180/\pi$ gives about 5.7106° . Then $0.7106 \times 60 \approx 42.636'$ and $0.636 \times 60 \approx 38.16''$: about $5^\circ 42' 38''$.

VERIFY BY HAND. The exact normalized point satisfies $100/101 + 1/101 = 1$. The angular estimate is an approximation, not an exact special angle.

Example 28 | A one-degree angle without a calculator

Estimate $\sin 1^\circ$, $\cos 1^\circ$, and $\tan 1^\circ$. Account for both the angle conversion and the series.

- 1. Convert first: $x = \pi/180$ radians. Using $3.14159 < \pi < 3.14160$ gives $0.017453277... < x < 0.017453334....$
- 2. For this small positive x , $x - x^3/6 < \sin x < x - x^3/6 + x^5/120$.
- 3. For cosine, $1 - x^2/2 < \cos x < 1 - x^2/2 + x^4/24$.
- 4. Pencil multiplication yields the safe rounded estimates below. Keep extra digits during arithmetic, then round at the end.

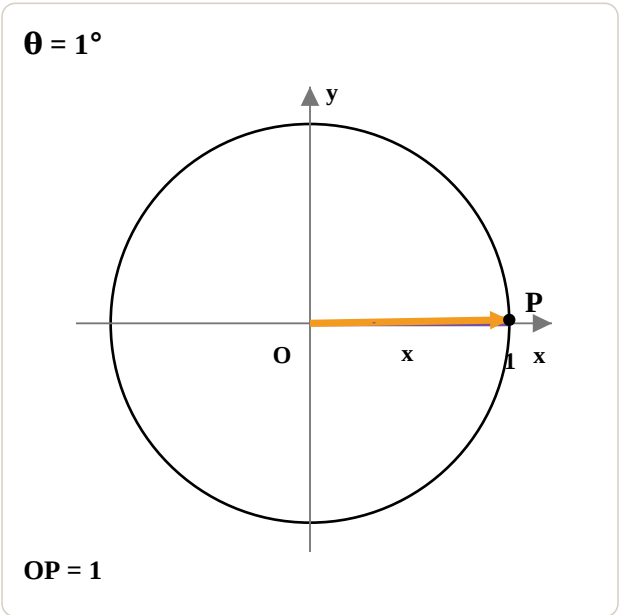


Figure 28. The orange arrow is the unit vector OP ; P is its head. Purple and blue legs show signed x and y projections.

DEGREES	RADIANS	TOTAL MINUTES	TOTAL SECONDS
1°	$\pi/180$	$60'$	$3600''$

$\sin \theta = y$	$\cos \theta = x$	$\tan \theta = y/x$
$\sin 1^\circ \approx 0.017452$	$\cos 1^\circ \approx 0.999848$	$\tan 1^\circ \approx 0.017455$

HEAD COORDINATES: $P = (\cos 1^\circ, \sin 1^\circ)$
UNIT VECTOR: $u = OP = (\cos 1^\circ, \sin 1^\circ)$; $\|u\| = 1$.

INVERSE CHECK. $\arcsin(\sin 1^\circ) = \arccos(\cos 1^\circ) = \arctan(\tan 1^\circ) = 1^\circ$ for the exact values.

KEY IDEA. Do not put the number 1 directly into the radian series for a one-degree angle. One radian is much larger than one degree.

VERIFY BY HAND. Safe intervals: $0.01745239 < \sin 1^\circ < 0.01745246$; $0.99984769 < \cos 1^\circ < 0.99984770$; $0.01745504 < \tan 1^\circ < 0.01745512$.

Chapter 7 | The complex circle and Euler’s identity

The plane with i

Write $z = x+iy$, where $i^2 = -1$. Its real part is x , its imaginary part is y , and its modulus is $|z| = \sqrt{x^2+y^2}$. Points with $|z| = 1$ form the same unit circle used throughout this book.

Why the exponential contains a circle

The exponential power series is $e^z = 1+z+z^2/2!+z^3/3!+\dots$. Substitute $z = i\theta$. Since the powers of i cycle through $1, i, -1, -i$, the even terms are real and the odd terms are imaginary. These series converge absolutely, so separating the two groups is valid.

Group the real and imaginary terms

$e^{i\theta} = (1-\theta^2/2!+\theta^4/4!-\dots) + i(\theta-\theta^3/3!+\theta^5/5!-\dots)$. The first parenthesis is $\cos \theta$; the second is $\sin \theta$. Hence $e^{i\theta} = \cos \theta + i \sin \theta$. The series use radians. This is Euler’s formula; the $\theta=\pi$ case is Euler’s identity.

Multiplication rotates; addition translates

$e^{iA}e^{iB} = e^{i(A+B)}$, so multiplying unit complex numbers adds their angles. Adding 1 has a different geometric effect: it shifts the point one unit to the right. In $e^{i\pi}+1$, the point -1 shifts exactly to the origin.

θ (RADIANs)	HEAD (x,y)	$e^{i\theta}$
0	(1,0)	1
$\pi/2$	(0,1)	i
π	(-1,0)	-1
$3\pi/2$	(0,-1)	-i
2π	(1,0)	1

Euler’s identity joins $e, i, \pi, 1$, and 0 through a precise geometric calculation. Zero is the resulting sum $-1+1$; the identity neither requires nor defines division by zero.

Distinguish the two operations in the identity: $e^{i\pi}$ makes a half-turn from 1 to -1 ; the final $+1$ translates that endpoint to 0 . The intermediate point has modulus 1 , while the final sum has modulus 0 . No contradiction occurs because addition need not preserve length.

Example 29 | A quarter-turn is multiplication by i

Interpret $e^{i\pi/2}$ on the complex plane and connect it to the unit vector.

- 1. A complex number $z = x+iy$ represents the point (x,y) . Here $i^2 = -1$.
- 2. Euler’s formula is $e^{i\theta} = \cos \theta + i \sin \theta$, with θ in radians.
- 3. At $\theta = \pi/2$, $\cos \theta = 0$ and $\sin \theta = 1$. Therefore $e^{i\pi/2} = 0+i = i$.
- 4. Multiplying $x+iy$ by i gives $-y+ix$, which maps (x,y) to $(-y,x)$: a counterclockwise quarter-turn.

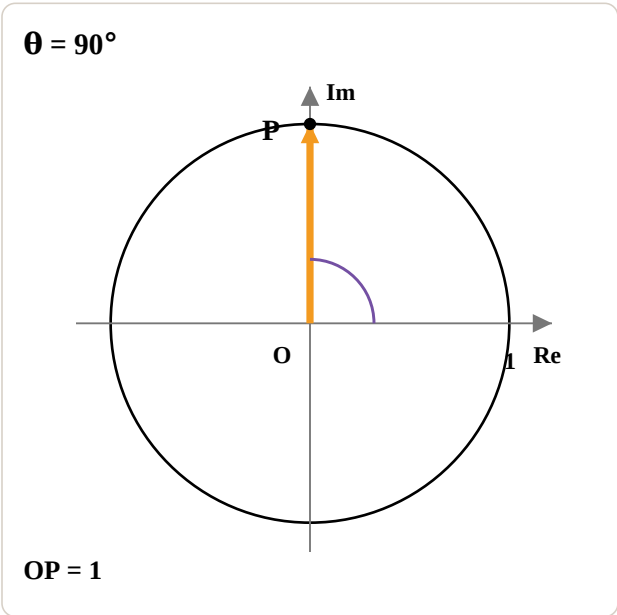


Figure 29. The orange arrow is the unit vector OP ; P is its head. At an axis angle, the reference triangle collapses.

DEGREES	RADIANS	TOTAL MINUTES	TOTAL SECONDS
90°	$\pi/2$	5400′	324000″

$\sin \theta = y$	$\cos \theta = x$	$\tan \theta = y/x$
1	0	undefined

HEAD COORDINATES: $P = (0, 1)$
UNIT VECTOR: $u = OP = (0, 1)$; $\|u\| = 1$.

INVERSE CHECK. $\arcsin(1) = \arccos(0) = \pi/2$. Tangent remains undefined even though the complex point i is well defined.

KEY IDEA. The horizontal and vertical coordinates become the real and imaginary parts. The geometry has not changed.

VERIFY BY HAND. The modulus is $|i| = \sqrt{0^2+1^2} = 1$. A pure rotation preserves vector length.

Example 30 | Compute $e^{i\pi} + 1 = 0$

Evaluate Euler’s identity exactly from the half-turn on the unit circle.

- 1. Start from $e^{i\theta} = \cos \theta + i \sin \theta$. The exponent angle is measured in radians.
- 2. Set $\theta = \pi$. A half-turn reaches $P = (-1,0)$, so $\cos \pi = -1$ and $\sin \pi = 0$.
- 3. Substitute: $e^{i\pi} = -1 + i(0) = -1$.
- 4. Add 1 to both sides: $e^{i\pi} + 1 = -1 + 1 = 0$. No decimal estimate of e or π is needed.

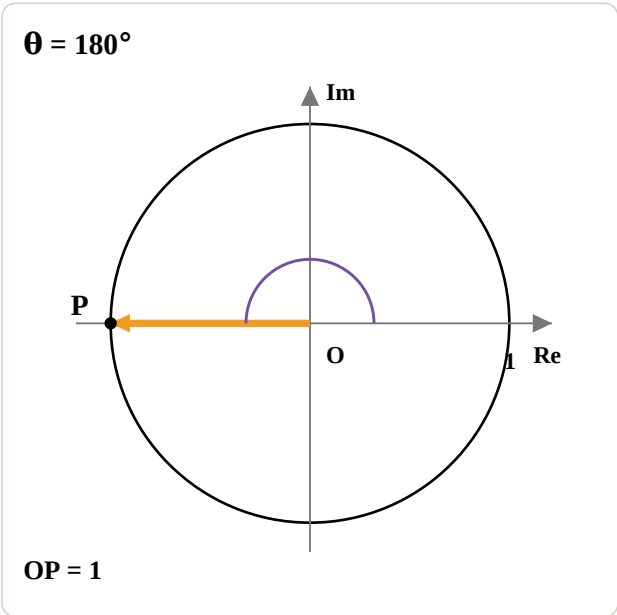


Figure 30. The orange arrow is the unit vector OP ; P is its head. At an axis angle, the reference triangle collapses.

DEGREES	RADIANS	TOTAL MINUTES	TOTAL SECONDS
180°	π	10800′	648000″

$\sin \theta = y$	$\cos \theta = x$	$\tan \theta = y/x$
0	-1	0

HEAD COORDINATES: $P = (-1, 0)$
UNIT VECTOR: $u = OP = (-1, 0)$; $\|u\| = 1$.

INVERSE CHECK. $\arcsin(0) = 0$; $\arccos(-1) = \pi$; $\arctan(0) = 0$. Only $\arccos$ returns the half-turn here.

KEY IDEA. $e^{i\pi}$ means the exponential evaluated at the complex input $i\pi$. It does not mean $e \times i \times \pi$.

VERIFY BY HAND. Another route: $e^{i\pi} = (e^{i\pi/2})^2 = i^2 = -1$. Adding 1 again gives 0.

Example 31 | Multiply rotations instead of measuring again

Square the complex unit point for 60° to find the point for 120° .

1. Let $z = 1/2 + i\sqrt{3}/2 = e^{i\pi/3}$. Multiplying two rotations adds their angles.
2. Square directly: $z^2 = (1/2)^2 + 2(1/2)(i\sqrt{3}/2) + (i\sqrt{3}/2)^2$.
3. Since $i^2 = -1$, the real part is $1/4 - 3/4 = -1/2$; the imaginary part is $\sqrt{3}/2$.
4. Thus $z^2 = -1/2 + i\sqrt{3}/2 = e^{i2\pi/3}$. Its coordinates match Example 9.

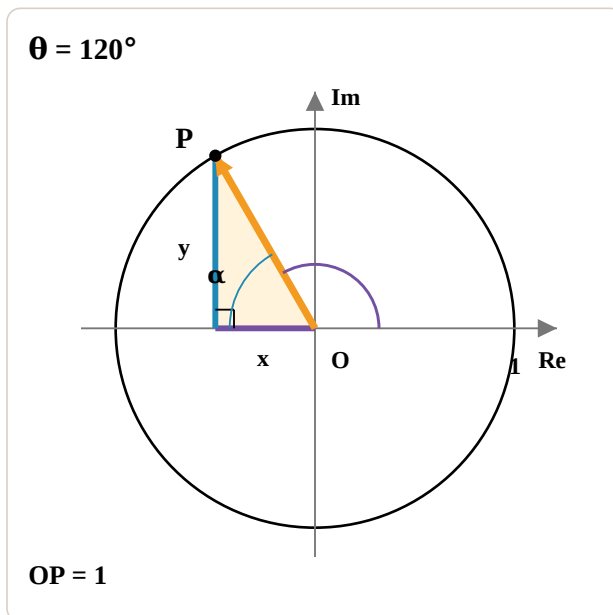


Figure 31. The orange arrow is the unit vector OP ; P is its head. Purple and blue legs show signed x and y projections.

DEGREES	RADIANS	TOTAL MINUTES	TOTAL SECONDS
120°	$2\pi/3$	$7200'$	$432000''$
$\sin \theta = y$	$\cos \theta = x$	$\tan \theta = y/x$	
$\sqrt{3}/2$	$-1/2$	$-\sqrt{3}$	

HEAD COORDINATES: $P = (-1/2, \sqrt{3}/2)$

UNIT VECTOR: $u = OP = (-1/2, \sqrt{3}/2)$; $\|u\| = 1$.

INVERSE CHECK. $\arccos(-1/2) = 2\pi/3$; $\arcsin(\sqrt{3}/2) = \pi/3$; $\arctan(-\sqrt{3}) = -\pi/3$.

KEY IDEA. Complex multiplication provides the same angle-addition laws as trigonometry. Unit moduli multiply to 1.

VERIFY BY HAND. $(e^{i\pi/3})^3 = e^{i\pi} = -1$. Six 60° rotations return to 1.

Example 32 | Three quarter-turns and the cycle of i

Use powers of i to find the point at 270° and explain why a full turn returns to 1.

- 1. Start at 1, the point (1,0). Each multiplication by i rotates by 90° counterclockwise.
- 2. $i^1 = i, i^2 = -1, i^3 = -i$, and $i^4 = 1$.
- 3. Three quarter-turns give $e^{i3\pi/2} = -i$, with head $P = (0, -1)$.
- 4. One more quarter-turn gives $e^{i2\pi} = 1$. The powers repeat with period 4.

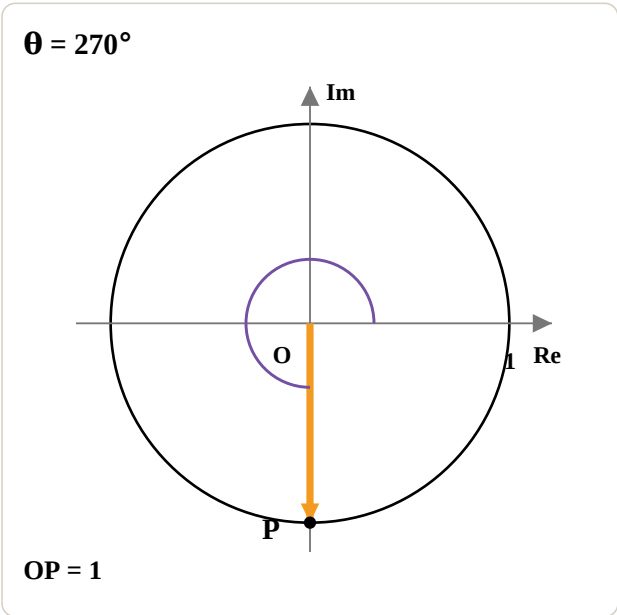


Figure 32. The orange arrow is the unit vector OP ; P is its head. At an axis angle, the reference triangle collapses.

DEGREES	RADIANS	TOTAL MINUTES	TOTAL SECONDS
270°	$3\pi/2$	$16200'$	$972000''$

$\sin \theta = y$	$\cos \theta = x$	$\tan \theta = y/x$
-1	0	undefined

HEAD COORDINATES: $P = (0, -1)$
UNIT VECTOR: $u = OP = (0, -1)$; $\|u\| = 1$.

INVERSE CHECK. $\arcsin(-1) = -\pi/2$; $\arccos(0) = \pi/2$. Neither records three positive quarter-turns.

KEY IDEA. Angles 270° and -90° have the same head. The complex number gives the final position, not a count of rotations.

VERIFY BY HAND. To simplify i^{2026} , divide 2026 by 4. The remainder is 2, so $i^{2026} = i^2 = -1$.

Reference atlas | one full turn

For every row, $P = (\cos \theta, \sin \theta)$ and u has those same components. Tangent is y/x . Use the degree column with the conversion table on the next page.

DEGREES	RADIANS	$\cos \theta = x$	$\sin \theta = y$	$\tan \theta$
0°	0	1	0	0
30°	$\pi/6$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}/3$
45°	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
60°	$\pi/3$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}$
90°	$\pi/2$	0	1	undefined
120°	$2\pi/3$	$-1/2$	$\sqrt{3}/2$	$-\sqrt{3}$
135°	$3\pi/4$	$-\sqrt{2}/2$	$\sqrt{2}/2$	-1
150°	$5\pi/6$	$-\sqrt{3}/2$	$1/2$	$-\sqrt{3}/3$
180°	π	-1	0	0
210°	$7\pi/6$	$-\sqrt{3}/2$	$-1/2$	$\sqrt{3}/3$
225°	$5\pi/4$	$-\sqrt{2}/2$	$-\sqrt{2}/2$	1
240°	$4\pi/3$	$-1/2$	$-\sqrt{3}/2$	$\sqrt{3}$
270°	$3\pi/2$	0	-1	undefined
300°	$5\pi/3$	$1/2$	$-\sqrt{3}/2$	$-\sqrt{3}$
315°	$7\pi/4$	$\sqrt{2}/2$	$-\sqrt{2}/2$	-1
330°	$11\pi/6$	$\sqrt{3}/2$	$-1/2$	$-\sqrt{3}/3$
360°	2π	1	0	0

How to rebuild this atlas: derive the 30° , 45° , and 60° values; place the four axis points; then reflect the acute points into the other quadrants. A full turn repeats the starting point.

Check any non-axis row by squaring x and y and adding. A negative coordinate has a positive square. Check tangent independently by dividing the displayed y -coordinate by the x -coordinate.

Reference atlas | arc units and new angles

DEGREES	RADIANS	TOTAL ARCMIN	TOTAL ARCSEC
0°	0	0′	0″
15°	$\pi/12$	900′	54000″
22.5°	$\pi/8$	1350′	81000″
30°	$\pi/6$	1800′	108000″
45°	$\pi/4$	2700′	162000″
60°	$\pi/3$	3600′	216000″
67.5°	$3\pi/8$	4050′	243000″
75°	$5\pi/12$	4500′	270000″
90°	$\pi/2$	5400′	324000″
120°	$2\pi/3$	7200′	432000″
135°	$3\pi/4$	8100′	486000″
150°	$5\pi/6$	9000′	540000″
180°	π	10800′	648000″
210°	$7\pi/6$	12600′	756000″
225°	$5\pi/4$	13500′	810000″
240°	$4\pi/3$	14400′	864000″
270°	$3\pi/2$	16200′	972000″
300°	$5\pi/3$	18000′	1080000″
315°	$7\pi/4$	18900′	1134000″
330°	$11\pi/6$	19800′	1188000″
360°	2π	21600′	1296000″

θ	$\cos \theta = x$	$\sin \theta = y$	$\tan \theta$
15°	$(\sqrt{6}+\sqrt{2})/4$	$(\sqrt{6}-\sqrt{2})/4$	$2-\sqrt{3}$
22.5°	$\sqrt{(2+\sqrt{2})}/2$	$\sqrt{(2-\sqrt{2})}/2$	$\sqrt{2}-1$
67.5°	$\sqrt{(2-\sqrt{2})}/2$	$\sqrt{(2+\sqrt{2})}/2$	$\sqrt{2}+1$
75°	$(\sqrt{6}-\sqrt{2})/4$	$(\sqrt{6}+\sqrt{2})/4$	$2+\sqrt{3}$

DMS reminders: $22.5^\circ = 22^\circ 30' 00''$ and $67.5^\circ = 67^\circ 30' 00''$. “Total arcminutes” includes all degrees converted into minutes; it is different from the minute field in a DMS expression.

Formula reference | choose a method

TASK	FORMULA OR METHOD
Normalize (a,b)	$r = \sqrt{a^2 + b^2}$; $u = (a/r, b/r)$, with $r > 0$.
Read the circle	$x = \cos \theta$; $y = \sin \theta$; $\tan \theta = y/x$ if $x \neq 0$.
Convert degrees	$\text{radians} = \text{degrees} \times \pi/180$.
Convert radians	$\text{degrees} = \text{radians} \times 180/\pi$.
Convert DMS	$\text{degrees} = D + M/60 + S/3600$ for a positive magnitude.
Compute arc length	$s = r\theta$, with θ in radians.
Pythagorean identity	$\sin^2 \theta + \cos^2 \theta = 1$.
Add or subtract sine	$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$.
Add cosine	$\cos(A+B) = \cos A \cos B - \sin A \sin B$.
Subtract cosine	$\cos(A-B) = \cos A \cos B + \sin A \sin B$.
Add tangent	$\tan(A+B) = (\tan A + \tan B)/(1 - \tan A \tan B)$, when defined.
Double an angle	$\sin 2A = 2\sin A \cos A$; $\cos 2A = \cos^2 A - \sin^2 A$.
Halve an angle	$\sin(A/2) = \pm \sqrt{[(1 - \cos A)/2]}$; $\cos(A/2) = \pm \sqrt{[(1 + \cos A)/2]}$.
Rational parameter	$x = (1 - t^2)/(1 + t^2)$, $y = 2t/(1 + t^2)$, $t = \tan(\theta/2)$.
Complex form	$e^{i\theta} = \cos \theta + i \sin \theta$; θ in radians.

Method chooser

Special acute angle? Build the 30° or 45° triangle. Another quadrant? Use a reference angle and signs. Sum or half of a known angle? Use an identity. Given coordinates? Normalize, then recover the quadrant. Non-special angle or inverse? Keep an exact symbolic expression and use a series with an error bound only when a decimal is required.

Every formula has a domain. Denominators must be nonzero, real square roots need nonnegative inputs, and half-angle signs depend on the quadrant. Series variables for trig functions are in radians.

Student exercises | A

Show the reasoning on separate paper. Use exact fractions and radicals unless a question requests an approximation. Give inverse-function answers in radians or clearly labeled degrees.

01. A right triangle has adjacent 12, opposite 5, hypotenuse 13. Find $\sin \theta$, $\cos \theta$, $\tan \theta$, and the unit-vector head.

02. Normalize $v=(8,15)$. Verify that the new vector has length 1.

03. Convert 45° into radians, total arcminutes, and total arcseconds.

04. Convert $7\pi/6$ radians into degrees, arcminutes, and arcseconds.

05. Convert $18^\circ 24' 36''$ into decimal degrees and an exact fraction of π radians.

06. Convert 12.375° into DMS.

07. A circle has radius 8 and arc length 4π . Find θ in radians and degrees.

08. Derive $\sin 45^\circ$ from a right triangle whose legs are both 1.

09. An equilateral triangle has side 10. Find its altitude, then $\sin 30^\circ$ and $\cos 30^\circ$.

10. At 270° , give the unit-vector head and explain the value of tangent.

11. Explain why $1'$ is not 0.01° and $1''$ is not 0.001° .

12. Convert $-22^\circ 30' 00''$ into decimal degrees and radians, with the minus sign applied to the entire angle.

Self-check routine: state the quadrant; verify $x^2+y^2=1$; compare the signs; check a principal inverse range; keep angle units visible. A diagram supports each solution even when no numerical approximation is needed.

Student exercises | B

Show the reasoning on separate paper. Use exact fractions and radicals unless a question requests an approximation. Give inverse-function answers in radians or clearly labeled degrees.

13. Find $\sin 150^\circ$, $\cos 150^\circ$, $\tan 150^\circ$, and P.
14. Find the three ratios at 225° .
15. Find the three ratios at -45° , then give a positive coterminal angle.
16. Find the head at 750° and the exact sine and cosine.
17. Evaluate $\arcsin(-\sqrt{3}/2)$, $\arccos(-1/2)$, and $\arctan(1)$.
18. Evaluate $\arcsin(\sin 150^\circ)$. Explain why it is not 150° .
19. Solve $\cos \theta = -1/2$ in $0^\circ \leq \theta < 360^\circ$. Give all real solutions too.
20. Solve $\tan \theta = -1$ in $0^\circ \leq \theta < 360^\circ$.
21. Derive $\tan 15^\circ$ using the difference formula and rationalize.
22. Use complementary angles to find $\sin 75^\circ$ from $\cos 15^\circ$.
23. Use $t=1/3$ in the rational half-angle parameter to find P and $\tan \theta$.
24. Normalize $\mathbf{v}=(-8,-15)$. State the quadrant and an exact angle in $[0,2\pi)$.

Self-check routine: state the quadrant; verify $x^2+y^2=1$; compare the signs; check a principal inverse range; keep angle units visible. A diagram supports each solution even when no numerical approximation is needed.

Student exercises | C

Show the reasoning on separate paper. Use exact fractions and radicals unless a question requests an approximation. Give inverse-function answers in radians or clearly labeled degrees.

25. Prove $1.73 < \sqrt{3} < 1.74$ by squaring. Then bound $\sqrt{3}/2$.
26. For $x=1/10$ rad, use $S=x-x^3/6$. State S and bound the next-term error.
27. For $x=1/10$ rad, use $C=1-x^2/2$. State C and bound the next-term error.
28. Use $\arctan z \approx z - z^3/3$ at $z=1/5$. Give a next-term bound.
29. Why is $\sin 1^\circ \approx 1$ an incorrect use of the small-angle formula?
30. Write $e^{(i\pi/4)}$ in $a+ib$ form and state its modulus.
31. Evaluate $e^{(i\pi)}+1$ by using sine and cosine.
32. Multiply i by $3+2i$. Interpret the resulting coordinate transformation.
33. Evaluate i^{37} and i^{100} .
34. Multiply $e^{(i\pi/6)}$ by $e^{(i\pi/3)}$. Give the angle and coordinates.
35. Explain why replacing π by 3.14 in $e^{(i\pi)}+1$ does not prove the exact identity.
36. A student says $\arcsin(1/2)=2$ because “inverse means reciprocal.” Correct the statement and name the reciprocal function.

Self-check routine: state the quadrant; verify $x^2+y^2=1$; compare the signs; check a principal inverse range; keep angle units visible. A diagram supports each solution even when no numerical approximation is needed.

Answer key | A

Use these results after completing the work. Equivalent exact forms are valid. The explanations identify the step that controls each answer.

01. $5/13$, $12/13$, $5/12$; $P=(12/13, 5/13)$. Divide all sides by 13.
02. Length 17; $u=(8/17, 15/17)$. The squared norm is $(64+225)/289=1$.
03. $\pi/4$ radians; $2700'$; $162000''$.
04. 210° ; $12600'$; $756000''$. Multiply by $180/\pi$ first.
05. $18+24/60+36/3600=18.41^\circ$. Radians $=1841\pi/18000$.
06. $12^\circ 22' 30''$: $0.375 \times 60 = 22.5$ minutes, and $0.5 \times 60 = 30$ seconds.
07. $\theta = s/r = \pi/2$ radians $= 90^\circ$.
08. Hypotenuse $\sqrt{2}$; $\sin 45^\circ = 1/\sqrt{2} = \sqrt{2}/2$.
09. Half-base 5; altitude $\sqrt{75} = 5\sqrt{3}$. Ratios: $1/2$ and $\sqrt{3}/2$.
10. $P=(0, -1)$; tangent is undefined because $y/x = (-1)/0$.
11. $1' = 1/60^\circ$ and $1'' = 1/3600^\circ$. Arc subdivisions use base 60.
12. $-22.5^\circ = -\pi/8$. The negative sign gives the direction of the whole angle.

Discuss the reasoning, not only the final value. A correct angle with the wrong unit, an unstated inverse branch, or an unjustified sign may hide a misunderstanding even when the number looks familiar.

Answer key | B

Use these results after completing the work. Equivalent exact forms are valid. The explanations identify the step that controls each answer.

13. $\sin=1/2$; $\cos=-\sqrt{3}/2$; $\tan=-\sqrt{3}/3$; $P=(-\sqrt{3}/2, 1/2)$.

14. $\sin=-\sqrt{2}/2$; $\cos=-\sqrt{2}/2$; $\tan=1$. The reference angle is 45° .

15. $\sin=-\sqrt{2}/2$; $\cos=\sqrt{2}/2$; $\tan=-1$. A positive coterminal angle is 315° .

16. $750^\circ - 720^\circ = 30^\circ$; $P=(\sqrt{3}/2, 1/2)$, sine $1/2$, cosine $\sqrt{3}/2$.

17. $-\pi/3$, $2\pi/3$, $\pi/4$ respectively, from the three principal ranges.

18. $\pi/6=30^\circ$. The principal arcsine range excludes 150° .

19. 120° or 240° ; all solutions $120^\circ + 360^\circ k$ or $240^\circ + 360^\circ k$, k an integer.

20. 135° and 315° . The complete family is $-45^\circ + 180^\circ k$.

21. $(1 - 1/\sqrt{3})/(1 + 1/\sqrt{3}) = (\sqrt{3} - 1)/(\sqrt{3} + 1) = 2 - \sqrt{3}$.

22. $\sin 75^\circ = \cos 15^\circ = (\sqrt{6} + \sqrt{2})/4$.

23. $x=(1 - 1/9)/(1 + 1/9)=4/5$; $y=(2/3)/(10/9)=3/5$; $\tan \theta=3/4$.

24. $u=(-8/17, -15/17)$, quadrant III; $\theta=\pi+\arctan(15/8)$.

Discuss the reasoning, not only the final value. A correct angle with the wrong unit, an unstated inverse branch, or an unjustified sign may hide a misunderstanding even when the number looks familiar.

Answer key | C

Use these results after completing the work. Equivalent exact forms are valid. The explanations identify the step that controls each answer.

25. $1.73^2 = 2.9929 < 3 < 3.0276 = 1.74^2$. Therefore $0.865 < \sqrt{3}/2 < 0.87$.

26. $S = 599/6000 = 0.0998333\dots$; $0 < \sin x - S < x^5/120 = 1/12000000$.

27. $C = 199/200 = 0.995$; $0 < \cos x - C < x^4/24 = 1/240000$.

28. $T = 74/375 = 0.197333\dots$ rad; $0 < \arctan(1/5) - T < 1/15625 = 0.000064$.

29. $\sin x \approx x$ requires radians. $1^\circ = \pi/180$ rad, about 0.01745 rad.

30. $\sqrt{2}/2 + i\sqrt{2}/2$, with modulus 1.

31. $\cos \pi + i \sin \pi + 1 = -1 + i(0) + 1 = 0$.

32. $i(3+2i) = -2+3i$; $(3,2) \rightarrow (-2,3)$, a 90° counterclockwise rotation.

33. 37 leaves remainder 1 on division by 4, so $i^{37} = i$. Since 100 is divisible by 4, $i^{100} = 1$.

34. The sum angle is $\pi/2$, so the product is i with coordinates $(0,1)$.

35. 3.14 is not exactly π ; its unit-circle point is not exactly $(-1,0)$. Use $\cos \pi = -1$ and $\sin \pi = 0$.

36. $\arcsin(1/2) = \pi/6$. The reciprocal function is cosecant: $\csc(\pi/6) = 1/\sin(\pi/6) = 2$.

Discuss the reasoning, not only the final value. A correct angle with the wrong unit, an unstated inverse branch, or an unjustified sign may hide a misunderstanding even when the number looks familiar.

Glossary | the language of the circle

TERM	DEFINITION
Adjacent leg	The non-hypotenuse side touching the selected acute angle.
Arc	A connected part of a circle; its length is measured along the curve.
Arcminute / arcsecond	1/60 of a degree and 1/3600 of a degree, respectively.
Argument	An angle describing a nonzero complex number's direction; angles can differ by full turns.
Coterminal	Having the same terminal ray after differing by whole revolutions.
Degree	1/360 of a complete revolution.
DMS	Degree-minute-second notation, using base 60 subdivisions.
Exact value	A value with no rounding error, such as $\sqrt{2}/2$ or $\arctan(3/4)$.
Head / tail	The end / start of a directed vector arrow.
Hypotenuse	The side opposite a right angle.
Inverse function	A function that reverses a one-to-one function on a specified domain.
Modulus	The length $ x+iy =\sqrt{(x^2+y^2)}$ of a complex number.
Normalize	Divide a nonzero vector by its length to obtain length 1.
Opposite leg	The side across from the selected acute angle.
Principal value	The single inverse output selected by a specified range.
Quadrant	One of the four open regions separated by the coordinate axes.
Radian	An angular unit based on $\theta=s/r$; a unit-radius arc of length 1 subtends 1 radian.
Reference angle	The acute angle from a non-axis terminal ray to the x-axis.
Terminal ray	The final ray after rotating from the initial ray.
Unit vector	A vector of length 1, used to describe a direction.

Notation: θ , α , β , and φ are angle variables; π is a constant; $i^2=-1$; k denotes an integer when describing whole-turn families. "rad" means radians. $P=(x,y)$ is a point; $u=(x,y)$ is a vector with those components.

Example index & subject index

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Suggested routes: First course: pages 3-25, then exercises A and B. Exact-value extension: pages 26-32. Hand computation: pages 33-37. Complex-number connection: pages 38-42. Keep pages 43-45 nearby as reference sheets.

Sources & classroom teaching notes

The explanations, exercise choices, layouts, and numbered diagrams in this book were prepared for this edition. The sources below support the standard mathematical definitions and methods. Accessed September 30, 2026.

1. OpenStax, Precalculus 2e, §5.1: Angles.

Degree-radian conversions, angular measure, and arc length.

<https://openstax.org/books/precalculus-2e/pages/5-1-angles>

2. OpenStax, Precalculus 2e, §5.2: Unit Circle: Sine and Cosine Functions.

Unit-circle coordinates and special-angle relationships.

<https://openstax.org/books/precalculus-2e/pages/5-2-unit-circle-sine-and-cosine-functions>

3. OpenStax, Precalculus 2e, §6.3: Inverse Trigonometric Functions.

Principal inverse ranges and recovering angles from ratios.

<https://openstax.org/books/precalculus-2e/pages/6-3-inverse-trigonometric-functions>

4. OpenStax, Precalculus 2e, Appendix A: Basic Functions and Identities.

Reference identities used in the exact-angle chapters.

<https://openstax.org/books/precalculus-2e/pages/a-basic-functions-and-identities>

5. Eric W. Weisstein, “Euler Formula,” MathWorld.

Complex exponential formula and the power-series route.

<https://mathworld.wolfram.com/EulerFormula.html>

6. Eric W. Weisstein, “Inverse Tangent,” MathWorld.

Arctangent, principal values, and series expansion.

<https://mathworld.wolfram.com/InverseTangent.html>

Teaching sequence

Ask students to derive one triangle, rebuild a quadrant table, and explain one inverse branch before using the atlas. Have partners check each other’s signs and units. For advanced groups, compare the geometry and power-series proofs of Euler’s formula.

Assessment and common errors

Look for unlabeled angle units, reversed coordinates, confusion between an inverse and a reciprocal, lost quadrant signs, and division by zero. Accept exact inverse expressions for non-special angles. Require error bounds when the task asks for a justified approximation.

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